lattice refinement in loop quantum cosmology

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loop quantum cosmology workshop

23-25 october 2008

outline

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motivation

cosmological predictions of quantum gravity

the inflationary paradigm provides a causal explanation for the primordial fluctuations with the correct features as measured in CMB

despite its successes, inflation has many shortcomings

- inflaton has not been measured in particle physics experiments
- the parameters of inflation need (too) often to be fine-tuned
- inflation is still a paradigm in search of a model
- inflation must prove itself generic

calzetta, sakellariadou 1991, 1992
gibbons, turok 2006
germani, nelson, sakellariadou 2007

in addition, there are questions that inflation does not address:

- what preceded inflation? --- the singularity is not resolved
- trans-Planckian problem

to address fundamental issues, we need a theory of quantum gravity

a quantum theory of gravity is expected to:

- cure classical singularities of GR
- provide information about initial conditions of the universe

⇨ either allow for onset of inflation, or provide an alternative

cosmology plays a dual role in quantum gravity, as:

- *a setting for QG to explain physical features of the universe*
- *a testing ground for any full theory of QG*

LQG: a nonperturbative and background independent canonical quantisation of GR in 4 space-time dimensions

LQC: a cosmological mini-superspace model quantised with methods of full LQG theory

LQC: SU(2) holonomies of the connection $\hat{p}$ & triads $\hat{h}_k$

- classical theory

curvature can be expressed as a limit of the holonomies around a loop as the area enclosed by the loop shrinks to zero

- quantum geometry

the loop cannot be continuously shrunk to zero area

the eigenvalues of the area operator are discrete

⇒ there is a smallest nonzero eigenvalue, the area gap Δ

the WDW equation gets replaced by a difference equation whose step size is controlled by Δ

isotropic models: $a(t)$

$$|\tilde{p}| = a^2$$

$$\tilde{c} = k + \gamma\dot{a}$$

triad component conjugate to the connection component

$$\{\tilde{c}, \tilde{p}\} = \frac{\kappa\gamma}{3}V_0$$

$$\kappa = 8\pi G$$

$$p = V_0^{2/3}\tilde{p}$$

$$c = V_0^{1/3}\tilde{c}$$

$$\{c, p\} = \frac{\kappa\gamma}{3}$$

old quantisation: follow procedure used in full LQG

ashtekar, bojowald, lewandowski 2003

$e^{i\mu_0 c/2}$, p : classical variables, with well-defined operator analogues

arbitrary real number

$$e^{i\mu_0 c/2} = \langle c|\mu\rangle$$

the eigenstates of $\hat{p}$ are the basis vectors $|\mu\rangle$:

$$\hat{p}|\mu\rangle = \frac{\kappa\gamma\hbar|\mu|}{6}|\mu\rangle$$

$$\widehat{\exp\left[\frac{i\mu_0}{2}c\right]}|\mu\rangle = \exp\left[\mu_0\frac{\mathrm{d}}{\mathrm{d}\mu}\right]|\mu\rangle = |\mu+\mu_0\rangle$$

the eigenstates of $\hat{p}$ satisfy the orthonormality condition:

$$\langle\mu_1|\mu_2\rangle = \delta_{\mu_1,\mu_2}$$

in the old quantisation, the operator $e^{i\mu_0 c/2}$ acts as a simple shift operator volume operator: $\widehat{V} = \widehat{|p|}^{3/2}$

volume of the elementary cell with eigenvalues: $V_\mu = \left(\frac{\kappa\gamma\hbar|\mu|}{6}\right)^{3/2}$

$$\hat{V}|\mu\rangle = \left(\frac{\kappa\gamma\hbar|\mu|}{6}\right)^{3/2}|\mu\rangle$$

$J = 1/2$

$$\widehat{V^{-1}}|\mu\rangle = \left|\frac{6}{\kappa\gamma\hbar\mu_0}\left(V_{\mu+\mu_0}^{1/3} - V_{\mu-\mu_0}^{1/3}\right)\right|^3|\mu\rangle$$

diagonal in $|\mu\rangle$ basis

proportional to the length of the holonomy

Hamiltonian constraint

the gravitational part of the Hamiltonian operator in terms of SU(2) holonomies and the triad component:

$$\hat{C}_{\rm grav} = \frac{2i}{\kappa^2\hbar\gamma^3\mu_0^3}{\rm tr}\sum_{ijk}\epsilon^{ijk}\left(\hat{h}_i^{(\mu_0)}\hat{h}_j^{(\mu_0)}\hat{h}i^{(\mu_0)-1}\hat{h}_j^{(\mu_0)-1}\hat{h}_k^{(\mu_0)}\left[\hat{h}_k^{(\mu_0)-1},\hat{V}\right]\right){\rm sgn}(\hat{p})$$

the holonomy along edge parallel to i th basis vector, of length $\mu_0 V_0^{1/3}$ w.r.t. fiducial metric

$$\hat{h}_i^{(\mu_0)} = \cos\widehat{\left(\frac{\mu_0 c}{2}\right)}\mathbb{1} + 2\sin\widehat{\left(\frac{\mu_0 c}{2}\right)}\tau_i$$

the identity 2x2 matrix

$$\tau_i = -i\sigma_1/2$$

a basis in the Lie algebra SU(2)

Pauli matrices

action of the self-adjoint Hamiltonian constraint operator

$\hat{\mathcal{H}}_{\rm g} = (\hat{\mathcal{C}}_{\rm g} + \hat{\mathcal{C}}_{\rm g}^{\dagger})/2$ on the basis states $|\mu\rangle$ is:

$$\hat{\mathcal{H}}_{\rm grav}|\mu\rangle = \frac{3}{4\kappa^2\gamma^3\hbar\mu_0^3}\Big\{[R(\mu)+R(\mu+4\mu_0)]|\mu+4\mu_0\rangle - 4R(\mu)|\mu\rangle + [R(\mu)+R(\mu-4\mu_0)]|\mu-4\mu_0\rangle\Big\}$$

$$R(\mu) = (\kappa\gamma\hbar/6)^{3/2}\Big||\mu+\mu_0|^{3/2} - |\mu-\mu_0|^{3/2}\Big|$$

dynamics are then determined by the Hamiltonian constraint:

$$(\hat{\mathcal{H}}_{\rm g} + \hat{\mathcal{H}}_{\phi})|\Psi\rangle \; = 0$$

full theory: infinite number of constraints
LQC: only one integrated Hamiltonian constraint

matter is introduced by adding the actions of matter components to the gravitational action
(just add the matter contribution to the Hamiltonian constraint)

obtain difference equations analogous to the differential WDW eqs the constraint equation:

$$\left[\left|V_{\mu+5\mu_0} - V_{\mu+3\mu_0}\right| + \left|V_{\mu+\mu_0} - V_{\mu-\mu_0}\right|\right]\Psi_{\mu+4\mu_0}(\phi) - 4\left|V_{\mu+\mu_0}V_{\mu-\mu_0}\right|\Psi_\mu(\phi)$$
$$+\left[\left|V_{\mu-3\mu_0} - V_{\mu-5\mu_0}\right| + \left|V_{\mu+\mu_0} - V_{\mu-\mu_0}\right|\right]\Psi_{\mu-4\mu_0}(\phi) = -\frac{4\kappa^2\gamma^3\hbar\mu_0^3}{3}\mathcal{H}_\phi(\mu)\Psi_\mu(\phi)$$

vandersloot 2005

$$|\Psi\rangle = \sum_\mu \Psi_\mu(\phi)|\mu\rangle$$

physical wave-functions

the matter Hamiltonian $\hat{\mathcal{H}}_\phi$ is assumed to act diagonally on the basis states with eigenvalues $\mathcal{H}_\phi$

quantum evolution equation

there is no continuous variable (the scale factor in classical cosmology)**, but a label with discrete steps**

the wave-function $\Psi_\mu(\phi)$ depending on internal time μ and matter fields ϕ determines the dependence of matter fields on the evolution of the universe

lattice refinement

old quantisation: quantised holonomies are fixed operators of fixed magnitude ⇒ instabilities in continuum semi-clssical limit

for a large semi-classical universe, the WDW wave-function would be oscillating on scales of order $(a\sqrt{\Lambda})^{-1}$

as the universe expands, this scale becomes eventually smaller than the discreteness scale of the difference equation of LQC

⇒ the discreteness of spatial geometry would become apparent in the behaviour of the wave-functions describing a classical universe

rosen, jung & khanna 2006

bojowald, cartin & khanna 2007

the form of the wave-functions indicates that the period of oscillations can decrease as the scale increases

⇨ at sufficiently large scales the assumption that the wave-functions are *pre-classical* breaks down

discrete continuous

Ψ *does not vary much on scales of* $4\mu_0$, so $\Psi_\mu(\phi) \approx \Psi(\mu, \phi)$

this would lead to QG corrections at large scale (classical) physics

to avoid this, was one of the motivations behind lattice refinement

bojowald & hinterleitner 2002

vandersloot 2005

allowing the length scale of the holonomies to vary, the form of the difference equation changes

assuming the lattice size is growing, the step-size of the difference equation is not constant in the original triad variables

the exact form of difference equation depends on lattice refinement

particular case: $\mu_0 \to \tilde{\mu}(\mu) = \mu_0 \mu^{-1/2}$

ashtekar, pawlowski, singh 2006

the basic operators are given by replacing μ_0 with $\tilde{\mu}$

upon quantisation $\widehat{e^{i\tilde{\mu}c/2}}|\mu\rangle = e^{-i\tilde{\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}}|\mu\rangle$

which is no longer a shift operator since $\tilde{\mu}$ is a function of μ

change the basis to:

$$\nu = \mu_0 \int \frac{\mathrm{d}\mu}{\tilde{\mu}} = \frac{2}{3}\mu^{3/2}$$

in the new variables the holonomies act as simple shift operators, with parameter length μ_0

$$e^{-i\tilde{\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}}|\nu\rangle = e^{-i\mu_0\frac{\mathrm{d}}{\mathrm{d}\nu}}|\nu\rangle = |\nu + \mu_0\rangle$$

$$\hat{V}|\nu\rangle = \frac{3\nu}{2}\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}|\nu\rangle$$

$$\hat{\mathcal{H}}_{\mathrm{g}}|\nu\rangle = \frac{9|\nu|}{16\mu_0^3}\left(\frac{\hbar}{6\kappa\gamma^3}\right)^{1/2}\left[\frac{1}{2}\Big\{U(\nu)+U(\nu+4\mu_0)\Big\}|\nu+4\mu_0\rangle - 2U(\nu)|\nu\rangle + \frac{1}{2}\Big\{U(\nu)+U(\nu-4\mu_0)\Big\}|\nu-4\mu_0\rangle\right]$$

vandersloot 2006

where $$U(\nu) = |\nu + \mu_0| - |\nu - \mu_0|$$

expand the general state in the kinematical Hilbert space in terms of the basis states $|\Psi\rangle = \sum_\nu \Psi_\nu(\phi)|\nu\rangle$

⇨ Hamiltonian constraint:

$$\frac{1}{2}\Big|\nu + 4\mu_0\Big|\Big[U(\nu + 4\mu_0) + U(\nu)\Big]\Psi_{\nu+4\mu_0}(\phi) + 2|\nu|U(\nu)\Psi_\nu(\nu) + \frac{1}{2}\Big|\nu - 4\mu_0\Big|\Big[U(\nu - 4\mu_0) + U(\nu)\Big]\Psi_{\nu-4\mu_0}(\phi)$$
$$= -\frac{16\mu_0^3}{9}\left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2}\mathcal{H}_\phi(\nu)\Psi_\nu(\phi)\ .$$

$\nu \gg \mu_0$ continuum limit of the Hamiltonian constraint in terms of μ

$$\mu^{-1/2}\frac{\partial}{\partial\mu}\left[\mu^{-1/2}\frac{\partial}{\partial\mu}\left(\mu^{3/2}\Psi(\mu,\phi)\right)\right] + \frac{8}{3}\left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2}\mathcal{H}_\phi(\mu)\Psi(\mu,\phi) + \mathcal{O}(\mu_0) + \ldots = 0$$

implications for inflation

nelson & sakellariadou PRD (2007) [0706.0179]

classically, the matter part of the Hamiltonian for a massive scalar field

momentum potential

$$\mathcal{H}_\phi = \kappa \left[\frac{P_\phi^2}{2a^3} + a^3 V(\phi) \right]$$

to quantise it use:

$$\hat{P}_\phi \Psi(p,\phi) = -i\hbar \frac{\partial \Psi(p,\phi)}{\partial \phi}$$

$$\hat{\phi}\Psi(p,\phi) = \phi \Psi(p,\phi)$$

in the continuum limit:

$$\hat{\mathcal{H}}_\phi \Psi(\mu,\phi) = -3 \left(\frac{6\hbar}{\kappa\gamma^3} \right)^{1/2} \mu^{-3/2} \frac{\partial^2 \Psi(\mu,\phi)}{\partial \phi^2} + \left(\frac{\kappa\gamma\hbar}{6} \right)^{3/2} \mu^{3/2} V(\phi) \Psi(\mu,\phi) + \mathcal{O}(\mu_0) + \cdots$$

in the large scale limit, the equation to be solved is:

$$p^{-1/2} \frac{\partial}{\partial p} \left[p^{-1/2} \frac{\partial}{\partial p} \left(p^{3/2} \Psi(p,\phi) \right) \right] + \beta V(\phi) p^{3/2} \Psi(p,\phi) = 0$$

$p = \kappa\gamma\hbar\mu/6$ $\beta = 96/(\kappa\hbar^2)$

approximate dynamics of the inflaton field by $V(\phi) = V_\phi p^{\delta-3/2}$

constant — slow-roll $\delta = 3/2$

by separation of variables: $\Psi(p,\phi) = \Upsilon(p)\Phi(\phi)$

in the large scale limit:

$$p^{-1/2}\frac{\mathrm{d}}{\mathrm{d}p}\left[p^{-1/2}\frac{\mathrm{d}}{\mathrm{d}p}\left(p^{3/2}\Upsilon(p)\right)\right] + \beta V_\phi p^\delta \Upsilon(p) = 0 \qquad \beta = \frac{96}{\kappa\hbar^2}$$

$$\Upsilon(p) \approx p^{-(9+2\delta)/8}\sqrt{\frac{2\delta+3}{2\sqrt{\beta V_\phi}\pi}}\left[C_1\cos\left(x - \frac{3\pi}{2(2\delta+3)} - \frac{\pi}{4}\right) + C_2\sin\left(x - \frac{3\pi}{2(2\delta+3)} - \frac{\pi}{4}\right)\right]$$

$$x = 4\sqrt{\beta V_\phi}(2\delta+3)^{-1}p^{(2\delta+3)/4}$$

integration constants

for the end of inflation to be describable using classical GR, it must end before a scale, at which the assumption of pre-classicality breaks down and the semi-classical description is no longer valid, is reached

the separation between two successive zeros of Υ_p is:

$$\Delta p \equiv \frac{\pi}{\sqrt{\beta V_\phi}} p^{(1-2\delta)/4}$$

for the continuum limit to be valid the wave-function must vary slowly on scales of the order of $4\tilde{\mu}$

so:

$$\Delta p > 4\mu_0 \left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2} p^{-1/2}$$

$$\tilde{\mu}(\mu) = \mu_0 \mu^{-1/2}$$

$$p = \kappa\gamma\hbar\mu/6$$

$$\Longrightarrow \quad V_\phi < \frac{27\pi^2}{192\mu_0^2\gamma^3\kappa^2\hbar} p^{(3-2\delta)/2}$$

set $\delta \approx 3/2$, $\mu_0 = 3\sqrt{3}/2$, $\gamma \approx 0.24$ $\hbar = 1$

$\Longrightarrow$ $$V(\phi) \lesssim 2.35 \times 10^{-2} l_{\rm Pl}^{-4}$$

whereas for fixed lattices: $V_\phi \ll 0.07 e^{-2N_{\rm cl}} l_{\rm pl}^{-4}$ if half of inflation takes place in classical era, then $V_\phi \ll 10^{-28} l_{\rm pl}^{-4}$

example: $V(\phi) = m^2\phi^2/2$ $\delta_H \approx 1.91 \times 10^{-5}$

$$\delta_{\rm H}^2(k) = \frac{1}{75\pi^2 M_{\rm Pl}^6} \frac{V^3(\phi)}{[V'(\phi)]^2}\Big|_{k=aH}$$

COBE-DMR CMB data: $\frac{[V(\phi)]^{3/2}}{V'(\phi)} \approx 5.2 \times 10^{-4} M_{\rm Pl}^3$ $\Longrightarrow$ $m\phi^2 \approx 1.5 \times 10^{-3} M_{\rm Pl}^3$

combining:

for lattice refinement: $m \lesssim 10\, M_{\rm Pl}$

for fixed lattices: $m \lesssim 70(e^{-2N_{\rm cl}})\, M_{\rm Pl}$ ← strong (fine-tuned) constraint on inflaton mass lattice refinement is necessary to achieve a natural inflationary model

lattice refinement and the matter Hamiltonian

nelson & sakellariadou PRD 76 (2007) 104003

assuming: $\tilde{\mu} = \mu_0 \mu^A$

then $\nu = \tilde{\mu}_0 \int \frac{\mathrm{d}\mu}{\tilde{\mu}(\mu)}$ leads to $\nu = \frac{\tilde{\mu}_0 \mu^{1-A}}{\mu_0(1-A)}$

Wheeler-De Witt constraint equation: $(\hat{\mathcal{H}}_{\rm g} + \hat{\mathcal{H}}_\phi)|\Psi\rangle = 0$

expanding in the $\nu \gg \tilde{\mu}_0$ limit and under the assumption of pre-classicality, the quantum constraint eq. becomes:

$$\frac{\partial^2 \Psi(\nu,\phi)}{\partial \nu^2} + \frac{\tilde{B}}{2\nu}\frac{\partial \Psi(\nu,\phi)}{\partial \nu} + \tilde{C}\nu^{-2}\Psi(\nu,\phi) + \beta \mathcal{H}_\phi \nu^{-B/2}\Psi(\nu,\phi) + \mathcal{O}(\tilde{\mu}_0) = 0$$

$$B = \frac{1-4A}{(1-A)}$$

$$\beta = \frac{\alpha^{3/2/(1-A)}}{12(1-A)^2}\left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2}$$

$$\tilde{B} = \frac{1-10A}{1-A},$$

$$\tilde{C} = \frac{(1+2A)(4A-1)+12A(2A-1)}{8(1-A)^2}$$

to solve the constraint equation we need the specific form of $\mathcal{H}_\phi$

large-scale limit: $\mathcal{H}_\phi = \epsilon(\phi)\nu^\delta$

in the large-scale limit the matter Hamiltonian can be approximated with:

$$\hat{\mathcal{H}}_\phi = \hat{\nu}^\delta \hat{\epsilon}\,(\phi)$$

constant w.r.t. ν

only valid for $\nu \gg 1$

$$\Longrightarrow \quad \hat{\epsilon}\,(\phi)\,\Psi \equiv \epsilon\,(\phi)\,\Psi = -\nu^{-\delta}\hat{\mathcal{H}}_{\text{grav}}\Psi$$

requirements for the wave-functions:

- ***normalisable solutions***

finite norm of physical wave-functions is conserved

$$\langle\Psi_1|\Psi_2\rangle_{\rm phys} = \int_{\phi=\phi_0} d\nu|\nu|^{\delta}\overline{\Psi}_1\Psi_2$$

⇨ the solutions of the constraint are normalisable provided they decay, on large scales, faster than $\nu^{-1/(2\delta)}$

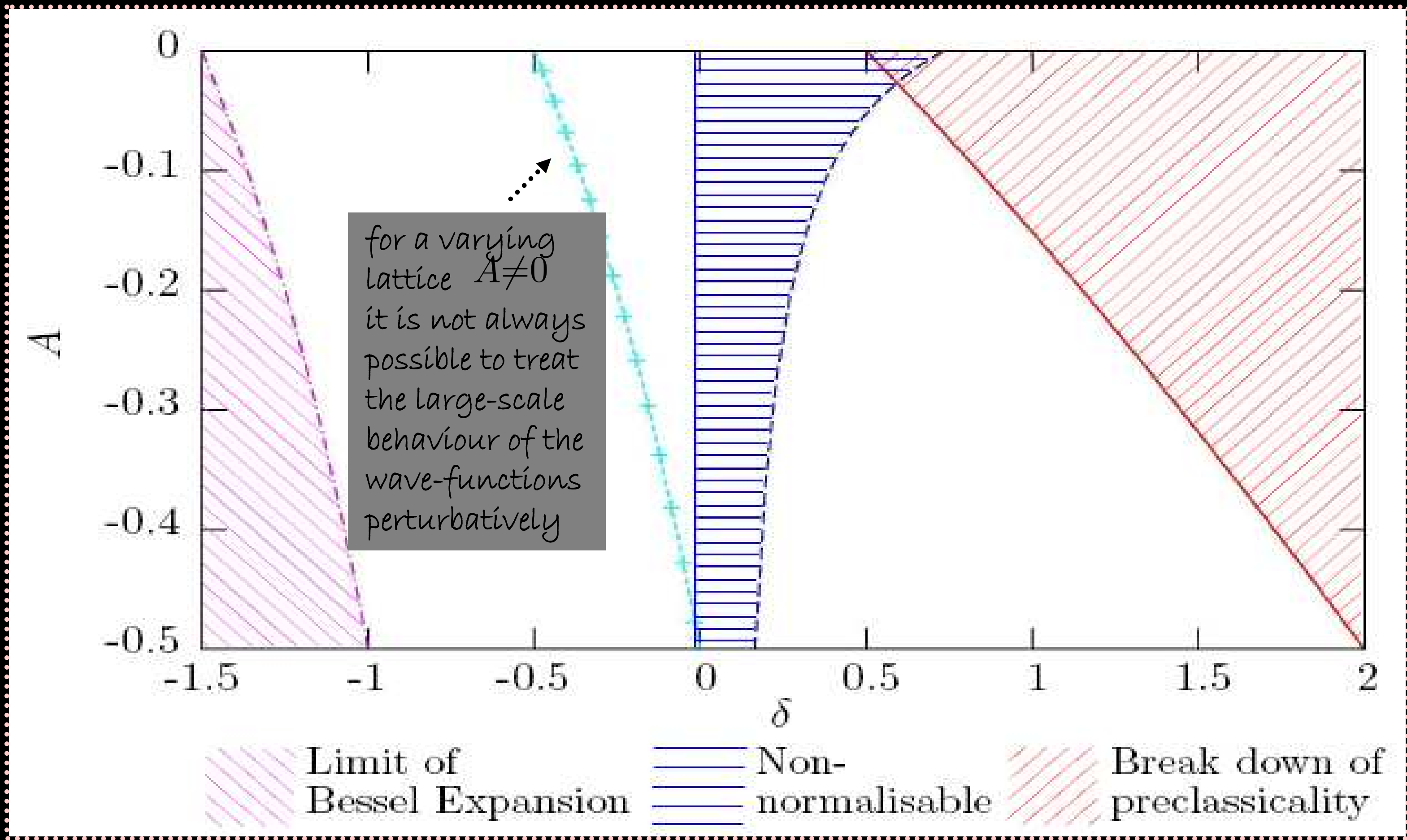
⇨ constraints on the scale dependence of matter component

- ***the solutions are valid on large scales, so the large argument expansions of Bessel functions should apply in this limit***
- ***the solutions should preserve pre-classicality***

⇨ constraints on 2-dim parameter space (A, δ)

full LQG theory allows only the range $0 < A < -1/2$

bojowald, cartin & khanna 2007



nelson & sakellariadou PRD 76 (2007) 104003 the continuum limit of the Hamiltonian constraint equation is sensitive to the choice of model and only a limited range of matter components can be supported within a particular choice

unique factor ordering in the continuum limit of LQC

nelson & sakellariadou PRD 78 (2008) 024006

Hamiltonian constraint:

$$\hat{C}_{\rm grav} = \frac{2i}{\kappa^2 \hbar \gamma^3 k^3} {\rm tr} \sum_{\rm ijk} \epsilon^{\rm ijk} \left(\hat{h}_{\rm i} \hat{h}_{\rm j} \hat{h}_{\rm i}^{-1} \hat{h}_{\rm j}^{-1} \hat{h}_{\rm k} \left[\hat{h}_{\rm k}^{-1}, \hat{V} \right] \right)$$

many possible choices of the factor ordering could have been made

each choice lead sto different factor ordering of continuum WDW

consider only factor ordering of the form of cyclic permutations of holonomy and volume operators with trace

find the action of different factor ordering choices

$$\tilde{\mu} = \mu_0 \mu^A \qquad \nu = k \int \frac{\mathrm{d}\mu}{\tilde{\mu}(\mu)} \qquad \nu = \frac{k\mu^{1-A}}{\mu_0(1-A)}$$

example:

consider $\epsilon_{\mathrm{ijk}} \mathrm{tr}\left(\hat{h}_{\mathrm{i}}\hat{h}_{\mathrm{j}}\hat{h}_{\mathrm{i}}^{-1}\hat{h}_{\mathrm{j}}^{-1}\hat{h}_{\mathrm{k}}\left[\hat{h}_{\mathrm{k}}^{-1}, \hat{V}\right]\right) = -24\widehat{\mathrm{Sn}}^2\widehat{\mathrm{Cs}}^2\left(\widehat{\mathrm{Cs}}\hat{V}\widehat{\mathrm{Sn}} - \widehat{\mathrm{Sn}}\hat{V}\widehat{\mathrm{Cs}}\right)$

$$\Rightarrow \quad \widehat{\mathrm{Sn}}^2\widehat{\mathrm{Cs}}^2\left(\widehat{\mathrm{Cs}}V\widehat{\mathrm{Sn}} - \widehat{\mathrm{Sn}}V\widehat{\mathrm{Cs}}\right)|\nu\rangle = \frac{-i}{32}\left(V_{\nu+k} - V_{\nu-k}\right)\Big(|\nu+4k\rangle - 2|\nu\rangle + |\nu-4k\rangle\Big)$$

extend it, to find action of particular factor ordering on a general state in Hilbert space $|\Psi\rangle = \sum_\nu \psi_\nu |\nu\rangle$

$$\Rightarrow \quad \epsilon_{\mathrm{ijk}} \mathrm{tr}\left(\hat{h}_{\mathrm{i}}\hat{h}_{\mathrm{j}}\hat{h}_{\mathrm{i}}^{-1}\hat{h}_{\mathrm{j}}^{-1}\hat{h}_{\mathrm{k}}\left[\hat{h}_{\mathrm{k}}^{-1}, \hat{V}\right]\right)|\Psi\rangle = \frac{-3i}{4}\sum_\nu\Big[\Big(V_{\nu-3k} - V_{\nu-5k}\Big)\psi_{\nu-4k} - 2\Big(V_{\nu+k} - V_{\nu-k}\Big)\psi_\nu + \Big(V_{\nu+5k} - V_{\nu+3k}\Big)\psi_{\nu+4k}\Big]|\nu\rangle$$

take continuum limit by expanding $\psi_\nu \approx \psi(\nu)$ as Taylor expansion in small k/ν [discreteness scale (k) << scale of universe (given by ν)]

large scale continuum limit of Hamiltonian constraint:

$$\lim_{k/\nu\to 0} \epsilon_{\rm ijk} {\rm tr}\left(\hat{h}_{\rm i}\hat{h}_{\rm j}\hat{h}_{\rm i}^{-1}\hat{h}_{\rm j}^{-1}\hat{h}_{\rm k}\left[\hat{h}_{\rm k}^{-1},\hat{V}\right]\right)|\Psi\rangle \sim$$
$$\frac{-36i}{1-A}\alpha^{3/[2(1-A)]}k^3\sum_\nu \nu^{(1+2A)/[2(1-A)]}\left[\frac{{\rm d}^2\psi}{{\rm d}\nu^2}+\frac{1+2A}{1-A}\frac{1}{\nu}\frac{{\rm d}\psi}{{\rm d}\nu}+\frac{(1+2A)(4A-1)}{(1-A)^2}\frac{1}{4\nu^2}\psi(\nu)\right]|\nu\rangle\ ,$$

continuum limit of WDW eq. $A=-1/2$

$$\lim_{k/\nu\to 0} \mathcal{C}_{\rm grav}|\Psi\rangle = \frac{72}{\kappa^2\hbar\gamma^3}\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\sum_\nu \frac{{\rm d}^2\psi}{{\rm d}\nu^2}|\nu\rangle$$

repeat the same analysis for all other factor orderings

for $A = -1/2$ all of them give the same continuum limit

using $\mu \sim p = a^2$ and $\nu \sim \mu^{3/2}$, the factor ordering of WDW eq. predicted by the large scale limit of LQC is:

$$\mathcal{C}_{\rm grav} \sim \frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} \sim a^{-2}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^{-2}\frac{\mathrm{d}\psi}{\mathrm{d}a}\right)$$

- phenomenological & consistency requirements indicate $A = -1/2$

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corichi & singh 2008

LQC predicts unique factor ordering of WDW eq. in its continuum limit

- require that factor ordering ambiguities in LQC disappear at level of WDW eq.

Lattice refinement model should be $A = -1/2$

numerical techniques in solving the quantum constraint equation of generic lattice-refinement models

there is a complication in directly evolving 2-dim wave-functions

the information needed to calculate the wave-function at a given lattice point is not provided by previous iterations

- local interpolation scheme to approximate the necessary data points

sabharwal & khanna 2007

- Taylor expansion to perform interpolation with well-defined and predictable accuracy

nelson & sakellariadou PRD78 (2008) 024030

1-dim system: a refined lattice can be mapped onto a fixed lattice by a change of basis

consider $$\tilde{\mu} = \mu_0 \mu^A$$

some constant

change of variables: $$\mu \to \nu = k \frac{\mu^{1-A}}{\mu_0 (1-A)}$$

a constant, equal to the magnitude of the shift operator associated with the new coordinates full Hamiltonian constraint on a constant lattice:

$$\frac{1}{k^3} S(\nu) \left[\Psi_{\nu+4k} - 2\Psi_\nu + \Psi_{\nu-4k}\right] = -\mathcal{H}_\phi$$

where

$$|\Psi\rangle = \sum_\nu \psi_\nu |\nu\rangle$$

$$S(\nu) = \left| |(\nu+k)\,\alpha|^{3/2/(1-A)} - |(\nu-k)\,\alpha|^{3/2/(1-A)} \right|$$

$$\alpha \equiv \frac{\mu_0(1-A)}{k}$$

of the same form as Hamiltonian constraint on a varying lattice:

$$\frac{1}{\tilde{\mu}^3} \left| |\mu+\tilde{\mu}|^{3/2} - |\mu-\tilde{\mu}|^{3/2} \right| \left[\Psi_{\mu+4\tilde{\mu}} - 2\Psi_\mu + \Psi_{\mu-4\tilde{\mu}}\right] = -\mathcal{H}_\phi$$

anisotropic geometry of black hole interior:

2-dim Hamiltonian constraint is a difference eq. on a varying lattice

$$
\begin{aligned}
&C_+(\mu,\tau)\left[\Psi_{\mu+2\delta_\mu,\tau+2\delta_\tau}-\Psi_{\mu-2\delta_\mu,\tau+2\delta_\tau}\right]\\
&+C_0(\mu,\tau)\left[(\mu+2\delta_\mu)\Psi_{\mu+4\delta_\mu,\tau}-2\left(1+2\gamma^2\delta_\mu^2\right)\mu\Psi_{\mu,\tau}+(\mu-2\delta_\mu)\Psi_{\mu-4\delta_\mu,\tau}\right]\\
&+C_-(\mu,\tau)\left[\Psi_{\mu-2\delta_\mu,\tau-2\delta_\tau}-\Psi_{\mu+2\delta_\mu,\tau-2\delta_\tau}\right]=\frac{\delta_\tau\delta_\mu^2}{\delta^3}\mathcal{H}_\phi\Psi_{\mu,\tau}\ ,
\end{aligned}
$$

bojowald, cartin & khanna 2007

$\delta_\mu(\mu,\tau)$

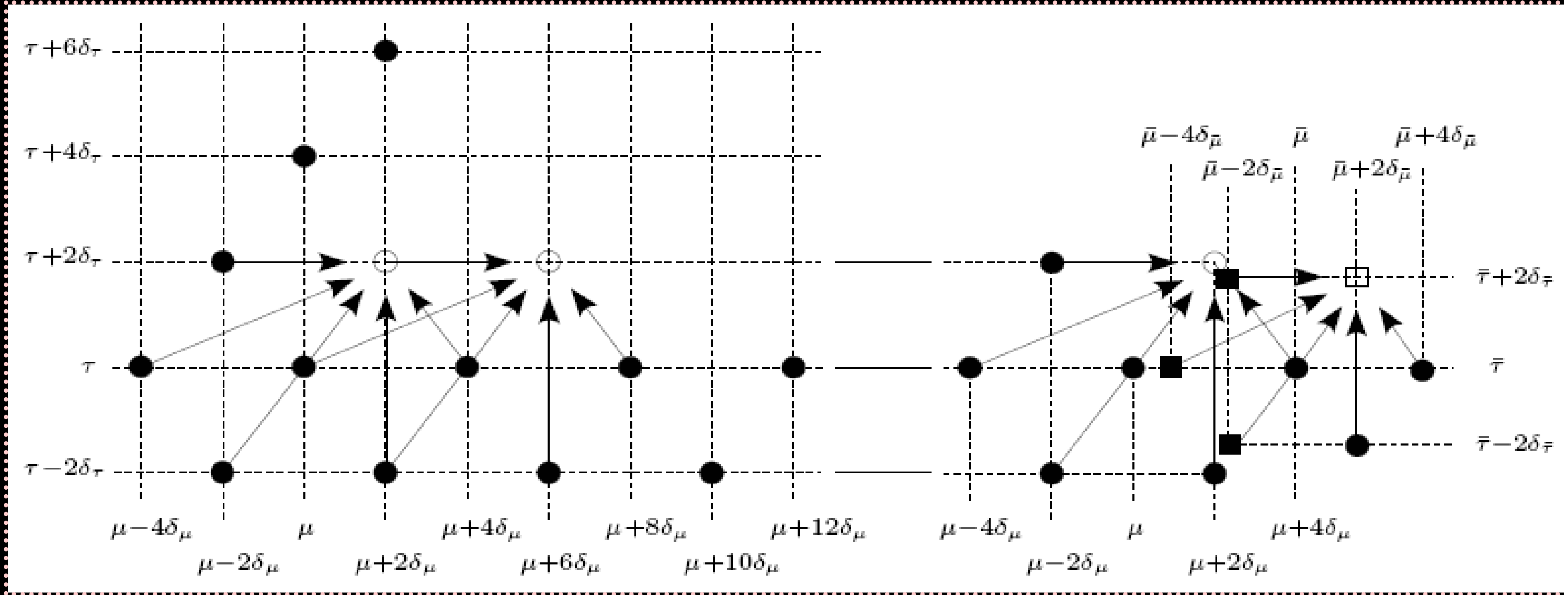
$\delta_\tau(\mu,\tau)$

$$
\begin{aligned}
C_\pm &\equiv 2\delta_\mu\left(\sqrt{|\tau\pm 2\delta_\tau|}+\sqrt{|\tau|}\right)\\
C_0 &\equiv \sqrt{|\tau+\delta_\tau|}-\sqrt{|\tau-\delta_\tau|}\ ,
\end{aligned}
$$

matter Hamiltonian acts diagonally on basis states of wavefunction

$$\hat{\mathcal{H}}_\phi|\Psi\rangle\equiv\hat{\mathcal{H}}_\phi\sum_{\mu,\tau}\Psi_{\mu,\tau}|\mu,\tau\rangle=\sum_{\mu,\tau}\mathcal{H}_\phi\Psi_{\mu,\tau}|\mu,\tau\rangle$$

$$\delta_{\bar{\mu}} \equiv \delta_\mu(\mu_{i+1}, \tau_i)$$
$$\delta_{\bar{\tau}} \equiv \delta_\tau(\mu_{i+1}, \tau_i)$$



for a fixed lattice the 2-dim wave-function can be calculated given suitable initial conditions (solid circles)

for a refining lattice, the data needed to calculate the value of the wave-function at a particular lattice site (open square) are not given by previous iterations (solid squares)

not constant → δ_μ, δ_τ decreasing functions of μ, τ

Taylor expansions to calculate the necessary data points:

given a function evaluated at three coordinates, the Taylor approximation to the value at a fourth position is:

$$f(x_4, y_4) = f(x_2, y_2) + \delta^x_{42} \left.\frac{\partial f}{\partial x}\right|_{x_2,y_2} + \delta^y_{42} \left.\frac{\partial f}{\partial y}\right|_{x_2,y_2} + \mathcal{O}\left((\delta^x_{42})^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_2,y_2} \right) + \mathcal{O}\left((\delta^y_{42})^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{x_2,y_2} \right)$$

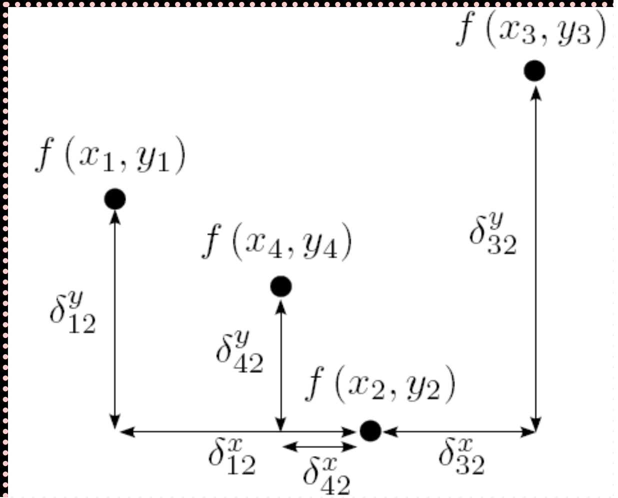
$$\delta^x_{ij} \equiv x_i - x_j$$

$$\delta^y_{ij} \equiv y_i - y_j$$

to approximate the differentials, use points (x_1, y_1) , (x_3, y_3)

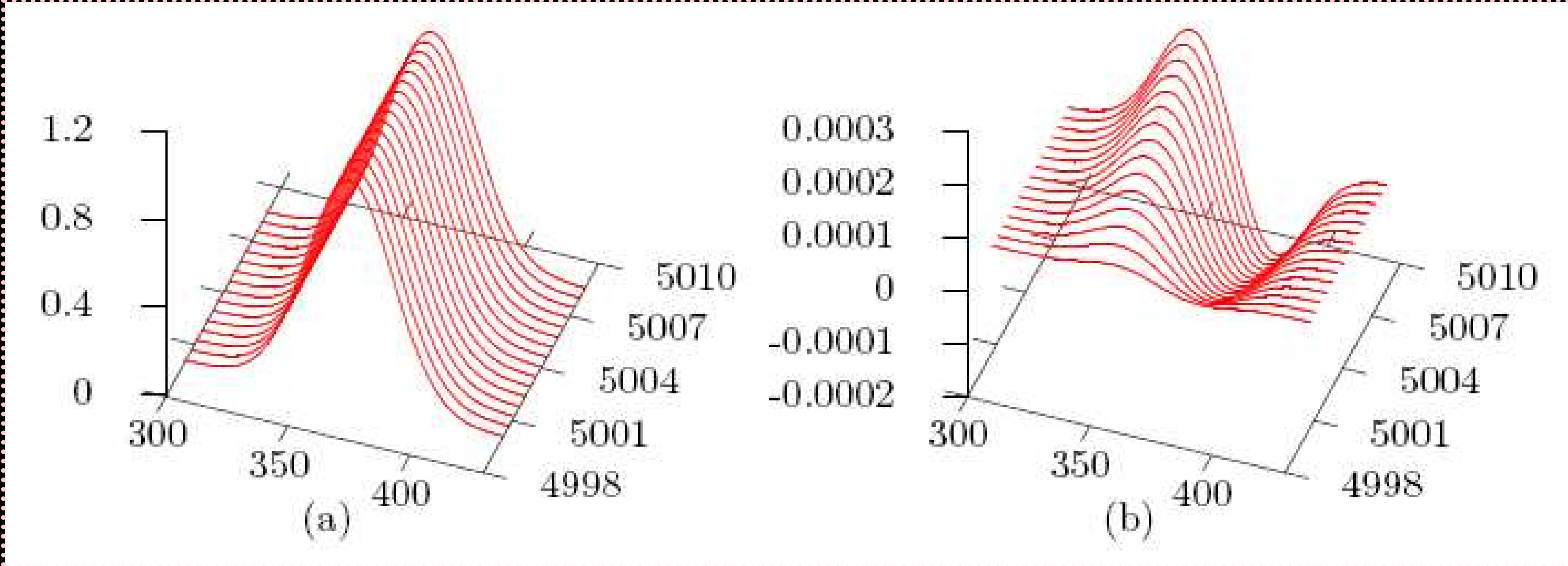
$$f(x_1, y_1) = f(x_2, y_2) + \delta^x_{12} \left.\frac{\partial f}{\partial x}\right|_{x_2,y_2} + \delta^y_{12} \left.\frac{\partial f}{\partial y}\right|_{x_2,y_2} + \cdots$$

$$f(x_3, y_3) = f(x_2, y_2) + \delta^x_{32} \left.\frac{\partial f}{\partial x}\right|_{x_2,y_2} + \delta^y_{32} \left.\frac{\partial f}{\partial y}\right|_{x_2,y_2} + \cdots$$



higher-order terms in Taylor expansion can be used to improve the accuracy of the system

for slowly varying wave-functions, the linear approximation is extremely accurate (higher-order corrections being $\approx 10^{-2}\%$)



the wave-function is calculated by iterating the difference eq. using 1st order Taylor expansion

2nd order correction

$$\delta_\mu(\mu, \tau) = \mu^{-1/2} \qquad \delta_\tau(\mu, \tau) = \tau^{-1/2}$$

stability of the Schwarszchild interior

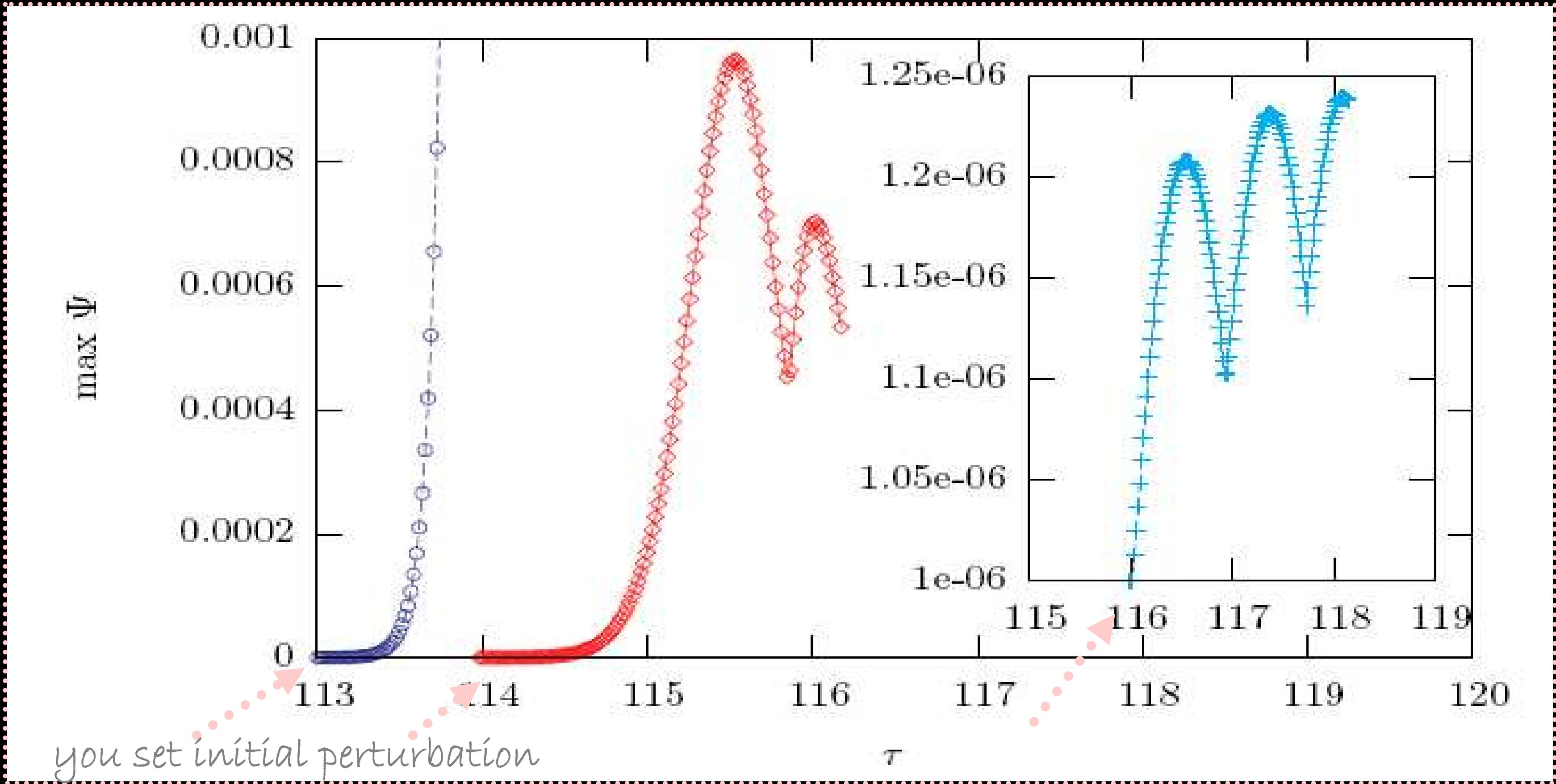
a von Neumann stability analysis of the difference equation

$$C_+(\mu,\tau)\left[\Psi_{\mu+2\delta_\mu,\tau+2\delta_\tau} - \Psi_{\mu-2\delta_\mu,\tau+2\delta_\tau}\right]$$
$$+C_0(\mu,\tau)\left[(\mu+2\delta_\mu)\Psi_{\mu+4\delta_\mu,\tau} - 2\left(1+2\gamma^2\delta_\mu^2\right)\mu\Psi_{\mu,\tau} + (\mu-2\delta_\mu)\Psi_{\mu-4\delta_\mu,\tau}\right]$$
$$+C_-(\mu,\tau)\left[\Psi_{\mu-2\delta_\mu,\tau-2\delta_\tau} - \Psi_{\mu+2\delta_\mu,\tau-2\delta_\tau}\right] = \frac{\delta_\tau\delta_\mu^2}{\delta^3}\mathcal{H}_\phi\Psi_{\mu,\tau}\ ,$$

for $\delta_\mu(\mu,\tau) = \mu_0\mu^{-1}$ $\delta_\tau(\mu,\tau) = \tau_0\tau^{-1}$

have shown that the system is unstable for $\mu > 2\tau$

bojowald, cartin & khanna 2007



a perturbation of $\Psi = 10^{-6}$ was put in an otherwise empty initial μ-row at $\mu = 230$ for the lattice refinement model $\delta_\mu(\mu,\tau) = \mu_0\mu^{-1}$ $\delta_\tau(\mu,\tau) = \tau_0\tau^{-1}$

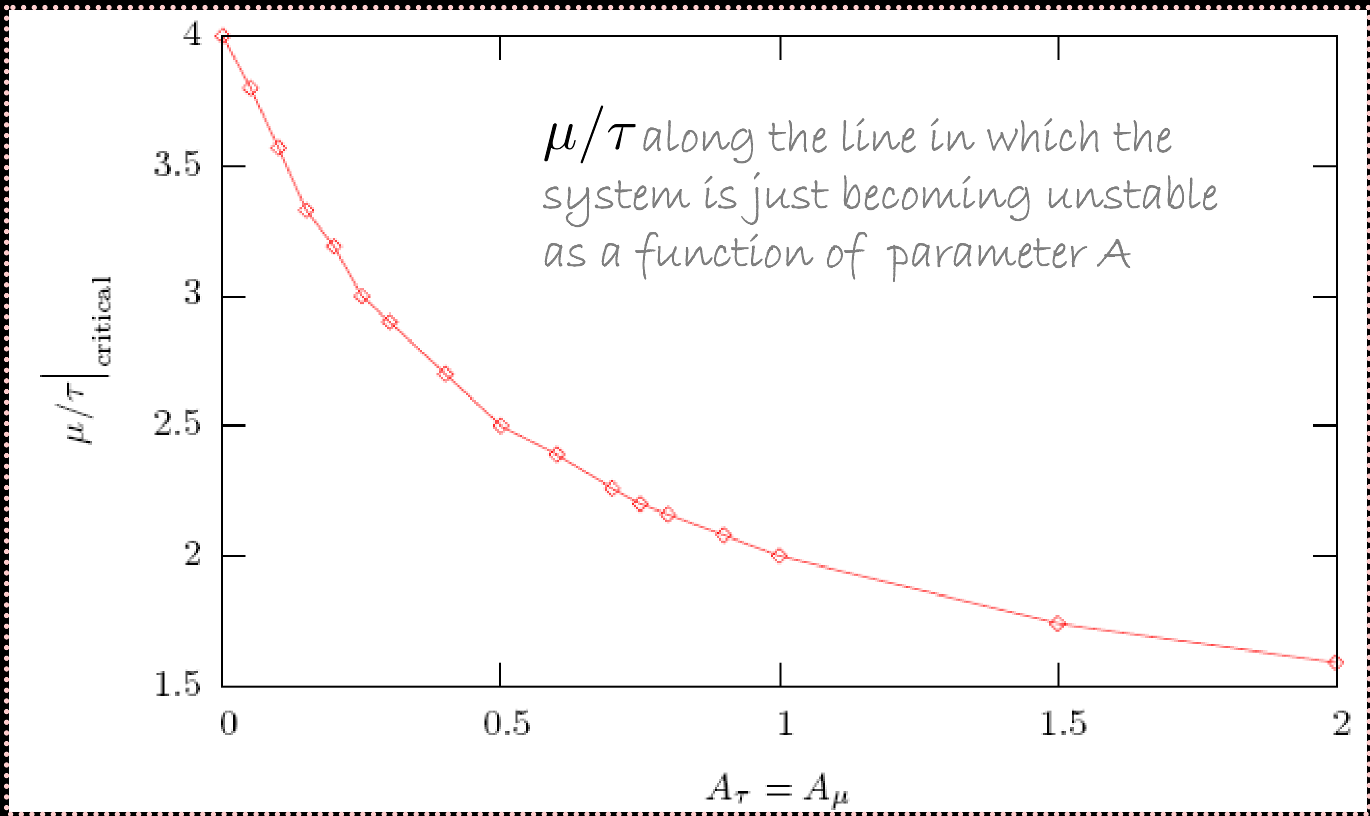
analytically, the region of stability is given for $\tau > \mu/2$, i.e., the amplitude of the perturbation grows exponentially for $\tau < 115$ and oscillates for $\tau > 115$. here we confirm this numerically

nelson & sakellariadou PRD78 (2008) 024030 investigate how lattice refinement can change the stability properties of the system

for $\delta_\mu = \mu^{-A}$ $\delta_\tau = \tau^{-A}$

the stability condition ranges between $\mu < 4\tau$ for $A = 0$ (constant lattice)

and $\mu < 1.58\tau$ for the case of $A = 2.0$



for $\delta_\mu(\mu,\tau) = \mu_0 \sqrt{\tau}\,\mu^{-1}$ $\delta_\tau(\mu,\tau) = \tau_0 \tau^{-1/2}$

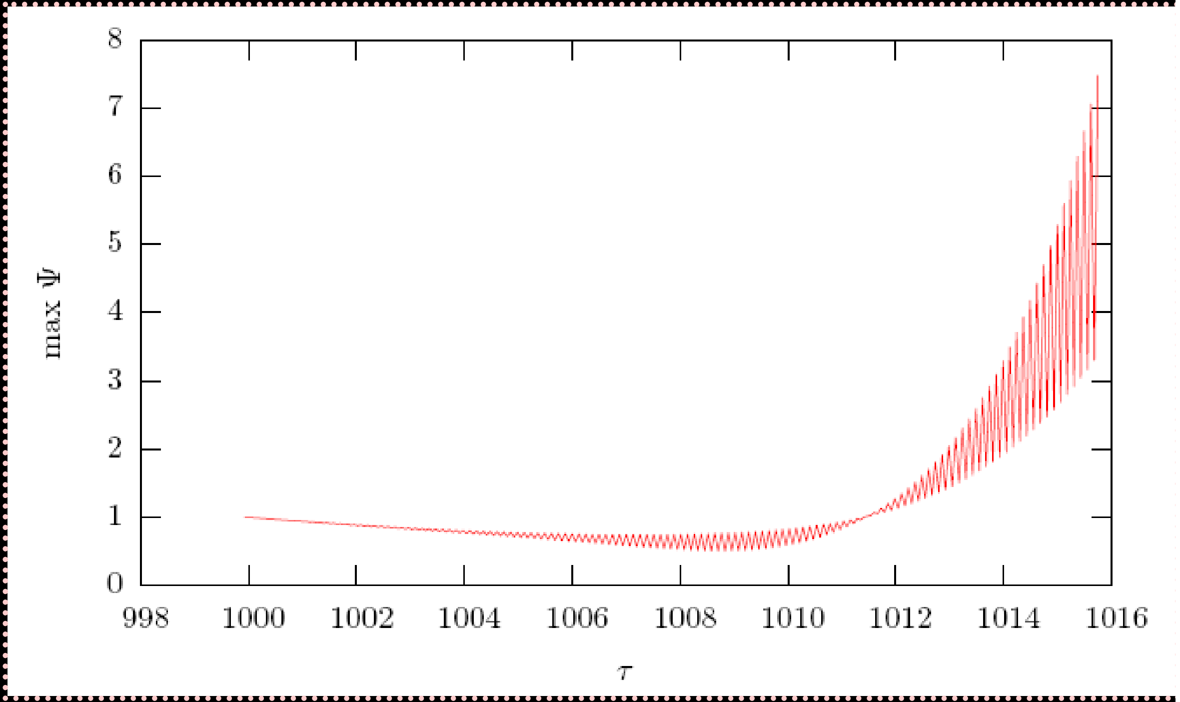
assuming the solutions do not change significantly on the scale of the step-sizes, the difference equation was found to be unconditionally stable

bojowald, cartin & khanna 2007

numerically this is indeed true, however as soon as the wave-functions fail to be pre-classical, they become unstable

a variation of about 0.1% between the wave-function in successive τ lattice points

the variation grows and when it reaches a few percent, the system becomes unstable



conclusions

LQC: quantised holonomies were taken to be shift operators with a fixed magnitud

the quantised Hamiltonian constraint is a difference equation with constant interval between points on the lattice

these models lead to serious instabilities in the continuum semi-classical limit

LQG: contributions to discrete Hamiltonian operator depend on the state which describes the universe

as the universe expands, the number of contributions increases, so the Hamiltonian constraint operator is expected to create new vertices of a lattice state, which in LQC result in a refinement of the discrete lattice

lattice refinement effect can be modelled and this approach eliminates problematic instabilities in continuum era

- lattice refinement seems to be necessary to achieve a natural inflationary model
- only a limited range of matter components can be supported within a particular choice
- factor ordering ambiguities in the continuum limit of the gravitational Part of Hamiltonian constraint disappear for a particular choice of lattice refinement

there is a complication in directly evolving 2-dim wave-functions, such as those necessary to study Bianchi models or black hole interiors

the information needed to calculate the wave-function at a given lattice point is not provided by previous iterations

Taylor expansions can be used to perform this interpolation with a well-defined and predictable accuracy

lattice refinement can change stability conditions of the system