Singularity Resolution in LQC: An Overview

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Contribution from many researchers, particularly:

Bojowald, Corichi, Chiou, Kaminski, Lewandowski, Mena

Pawlowski Singh, Szulc, Taveras, Vandersloot, Wilson-Ewing, Willis, ...

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- In general relativity, the gravitational field encoded in the very geometry of space-time $\Rightarrow$ space-time itself ends at singularities. General expectation: theory is pushed beyond its domain of applicability. Must incorporate quantum physics. Singularities are our gateways to physics beyond Einstein.

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- But straightforward incorporation of quantum physics a la traditional WDW quantum cosmology did not succeed.

- Situation very different in LQG: Physics does not stop at these singularities. Quantum Geometry extends its life. Resolution of space-like singularities has been analyzed at three levels:

i) Quantum Hamiltonian constraint does not break down. (Cosmological and black hole interior models, some midi-superspaces)

ii) + construction of the Physical Hilbert space, Dirac observables, emergent time (homogeneous models);

iii) + Detailed numerical solutions, effective equations and comparison between the two; exactly soluble models (homogeneous, isotropic). Important questions (raised by Brunnemann & Thiemann; Green & Unruh) have been answered.

- Goal: To present an overview and discuss conceptual issues, emphasizing subtleties and clarifying the overall situation to set the stage. Can be confusing because, as the subject evolved, we learned from mistakes. Will present the final picture for rather than summary of how things evolved.

- In LQC three tools have been used:

i) Numerical evolution in exact LQC
ii) Effective equations
iii) Exact analytical results in the k=0, Λ=0 model

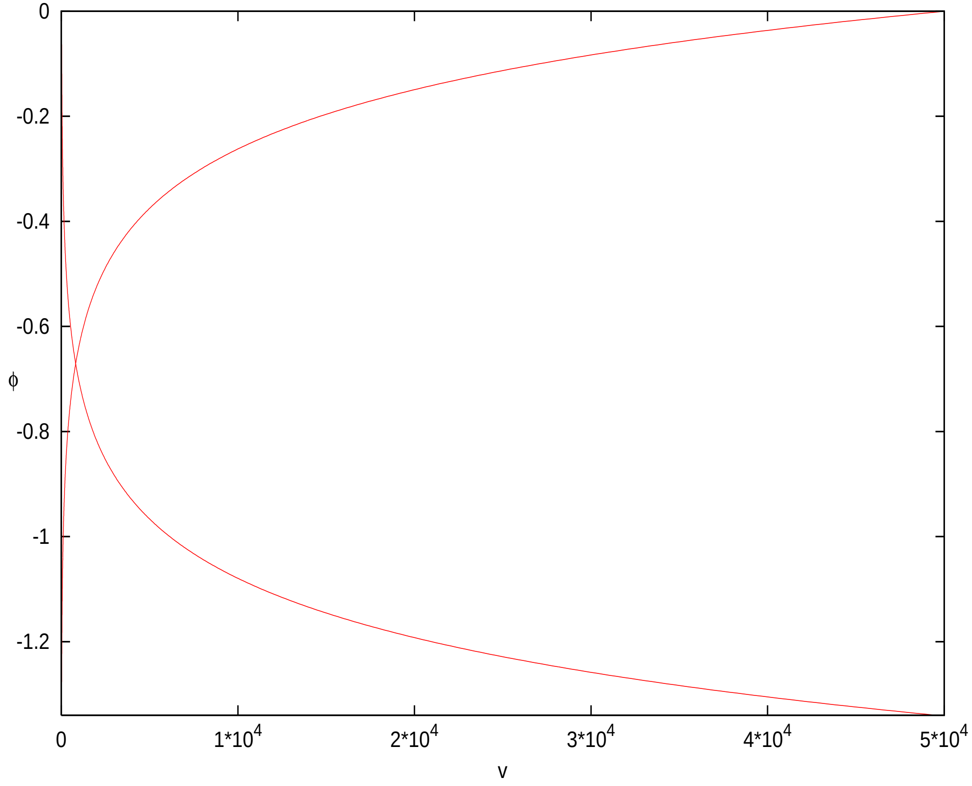
- Organization:

I. Summary of the FRW Models in LQC
II. FAQs and conceptual clarifications.

(Oriented toward discussion.)

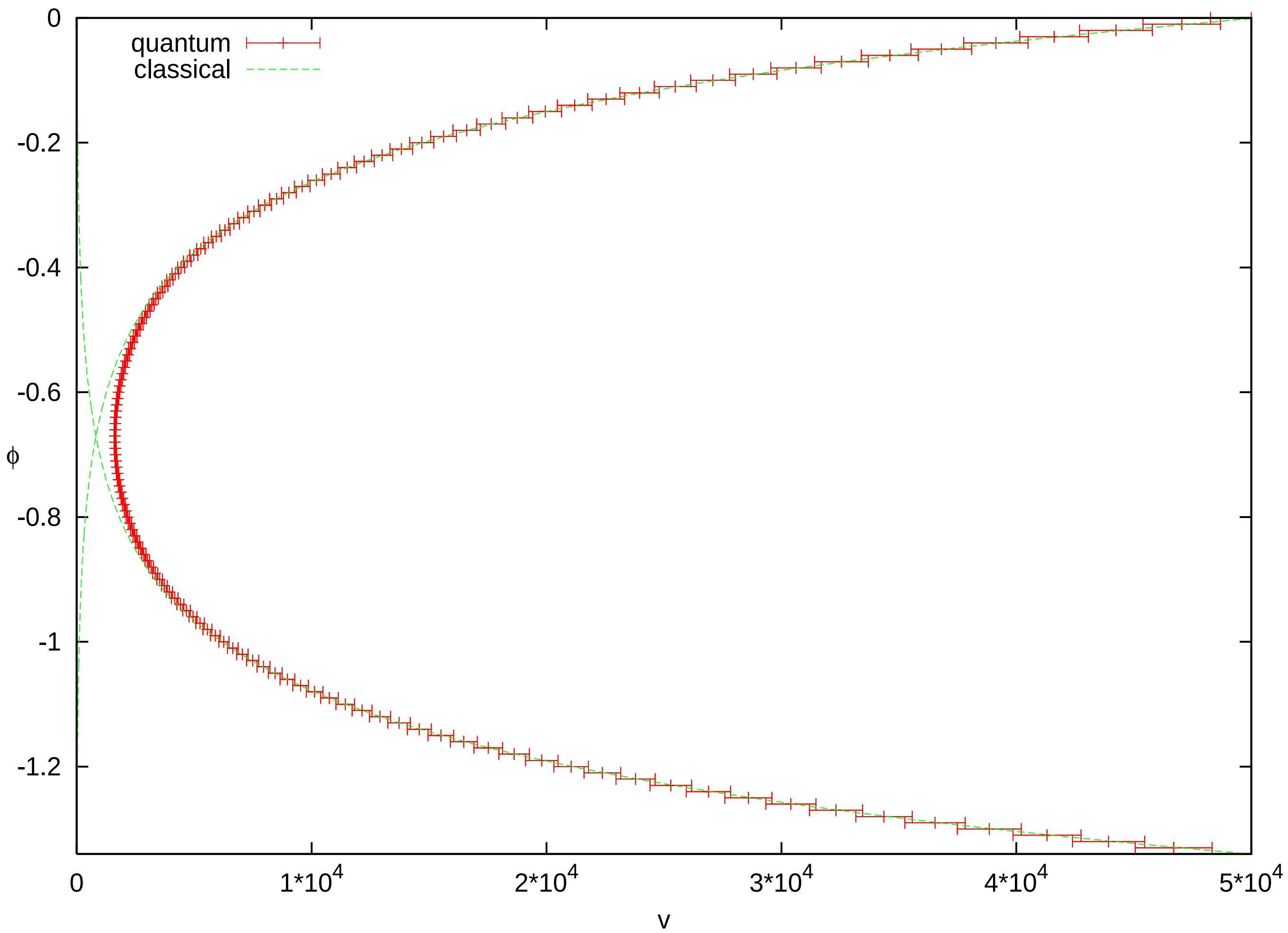
I.1. k=0 Models

FRW, k=0 Model coupled to a massless scalar field ϕ. Instructive because every classical solution is singular. Provides a foundation for more complicated models.



Classical trajectories

k=0 LQC: Gamow's Preference Realized!



Expectations values and dispersions of $\hat{V}|_{\phi}$ & classical trajectories.

k=0 Results

Assume that the quantum state is semi-classical at a late time and evolve backwards and forward. Then: (AA, Pawlowski, Singh)

• The state remains semi-classical till *very* early and *very* late times, i.e., till $R \approx 1/lp^2$ or $\rho \approx 0.01\rho_{\rm Pl}$. $\Rightarrow$ We know 'from first principles' that space-time can be taken to be classical during the inflationary era (since $\rho \sim 10^{-12}\rho_{\rm Pl}$ at the onset of inflation).

• In the deep Planck regime, semi-classicality fails. But quantum evolution is well-defined through the Planck regime, and remains deterministic unlike in other approaches. No new principle needed.

• The situation is the same if we include a cosmological constant (Pawlowski's talk) or an inflationary potential (AA, Pawlowski, Singh).
In all cases, the quantum space-time is vastly larger than what general relativity had us believe.

k=0 Results

• No unphysical matter. All energy conditions satisfied. But the left side of Einstein's equations modified because of quantum geometry effects (discreteness of eigenvalues of geometric operators.): Main difference from WDW theory.

• To compare with the standard Friedmann equation, convenient to do an algebraic manipulation and move the quantum geometry effect to the right side. Then:
$(\dot{a}/a)^2 = (8\pi G\rho/3)[1 - \rho/\rho_{\rm crit}]$ where $\rho_{\rm crit} \sim 0.41\rho_{\rm Pl}$.
Big Bang replaced by a quantum bounce.

• The matter density operator $\hat\rho = \frac{1}{2}\,(\hat{V}_\phi)^{-1}\,\hat{p}^2_{(\phi)}\,(\hat{V}_\phi)^{-1}$ has an absolute upper bound on the physical Hilbert space (AA, Corichi, Singh):

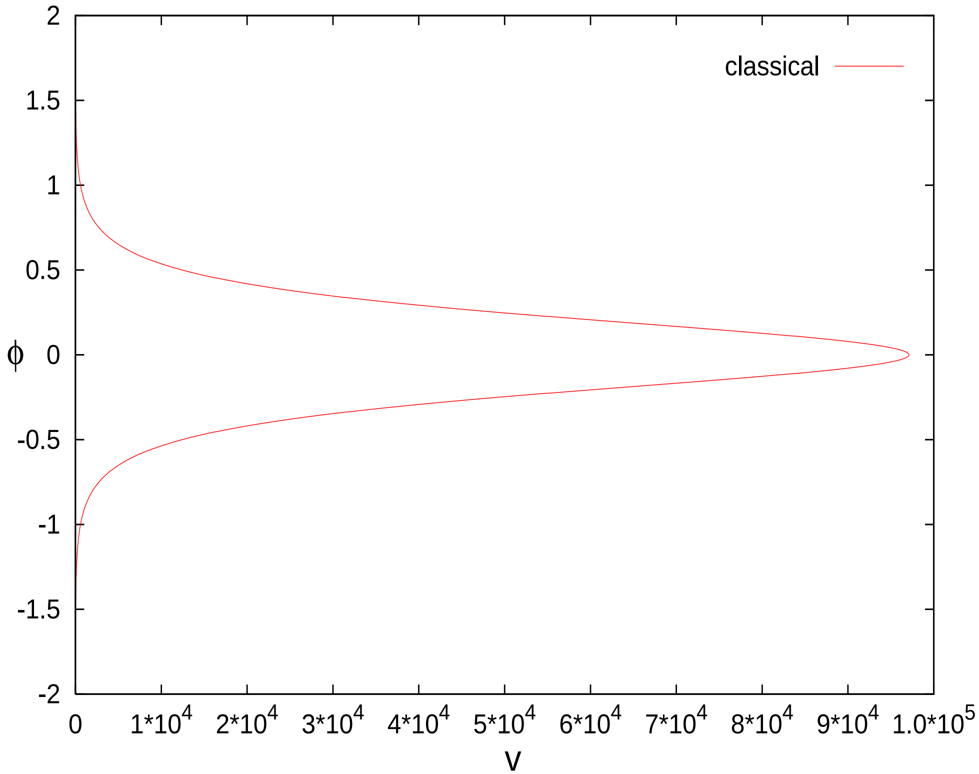
$$\rho_{\rm sup} = \sqrt{3}/16\pi^2\gamma^3 G^2\hbar \approx 0.41\rho_{\rm Pl}!$$

Provides a precise sense in which the singularity is resolved.

• Quantum geometry creates a brand new repulsive force in the Planck regime, replacing the big-bang by a quantum bounce. Physics does not end at singularities.

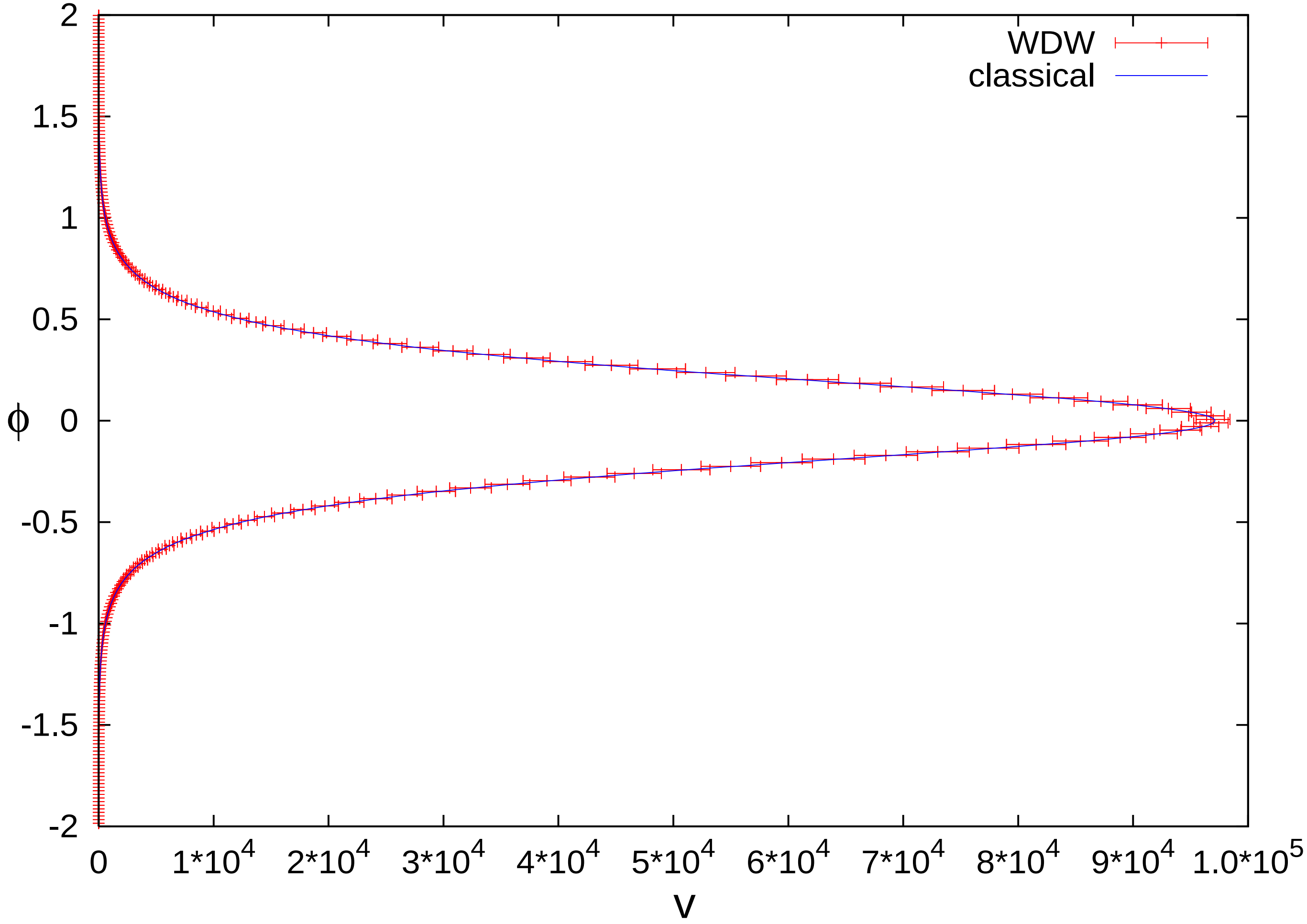
I.2 The Closed Model

Another Example: $k = 1$ FRW model with a massless scalar field ϕ. Instructive because again every classical solution is singular; scale factor not a good global clock; More stringent tests because of the classical re-collapse. (de Sitter, Tolman, Sakharov, Dicke,...)



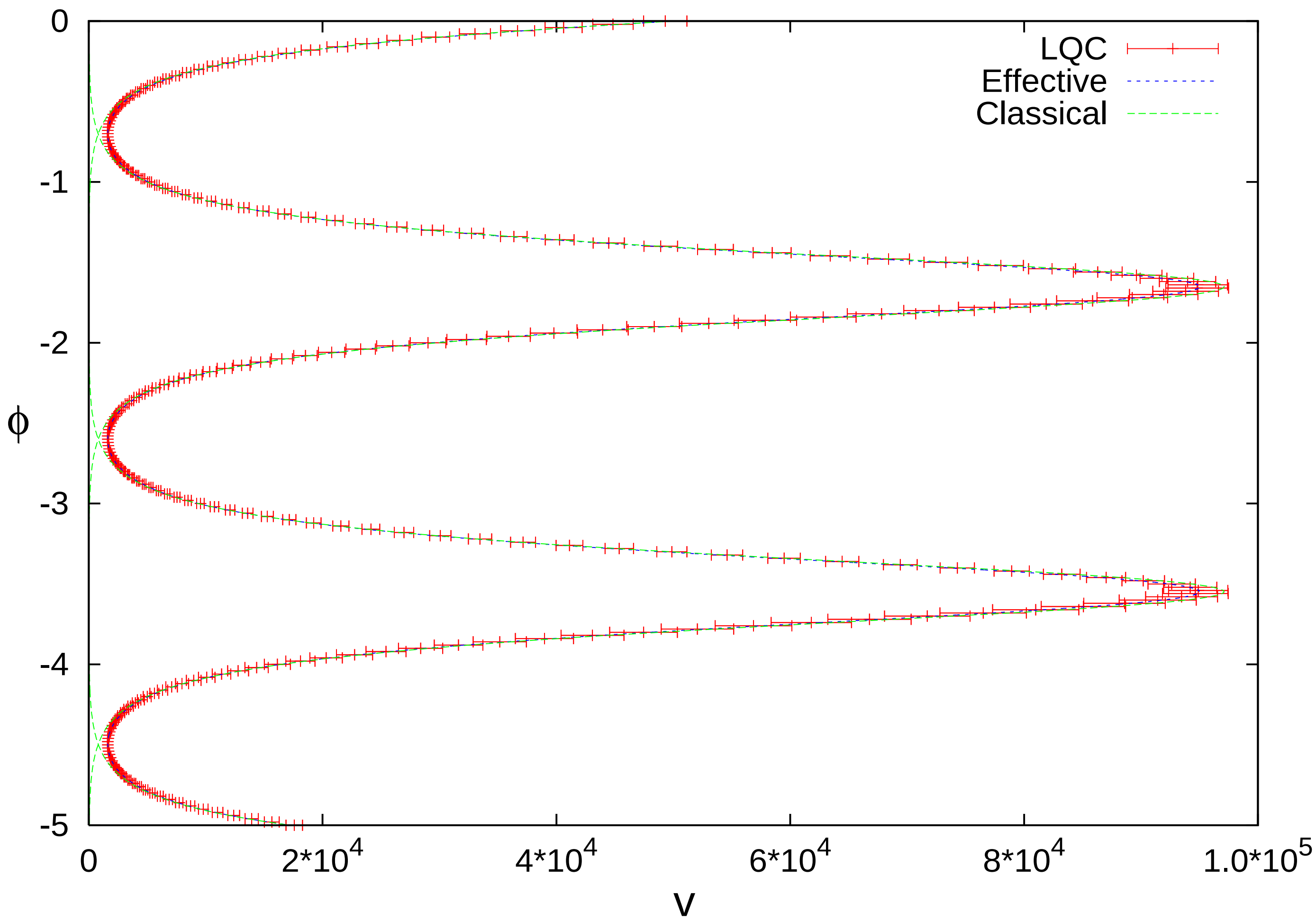
Classical Solutions

k=1 Model: WDW Theory



Expectations values and dispersions of $\hat{V}|_{\phi}$.

k=1 LQC: : Bouncing/Phoenix Universes.



Expectations values and dispersions of $\hat{V}|_{\phi}$ & classical trajectories.

(AA, Pawlowski, Singh, Vandersloot)

k=1: Key Results

(AA, Pawlowski, Singh, Vandersloot)

- Classical Re-collapse: The infra-red issue.

$$\rho_{\min} = (3/8\pi G a_{\max}^2)\left(1 + O({\ell_{\rm pl}}^4/a_{\max}^4)\right)$$

So, even for a very small universe, $a_{\max} \approx 23\ell_{\rm pl}$, (i.e. $p_{(\phi)} = 5 \times 10^3\hbar$), agreement with the classical Friedmann formula to one part in 10^5. Classical GR an excellent approximation between $a \sim 8\ell_{\rm pl}$ and $a \sim 23\ell_{\rm pl}$. For macroscopic universes, LQC prediction on recollapse indistinguishable from the classical Friedmann formula.

- Quantum Bounces: The ultra-violet issue

For a universe which attains $v_{\max} \approx 1\,\mathrm{Mpc}^3$, $v_{\min} \approx 6 \times 10^{16}\mathrm{cm}^3 \approx 10^{115}{\ell_{\rm pl}}^3$! What matters is curvature which enters Planck regime at this volume.

I.3 Summary

• In LQG, the interplay between geometry and physics is elevated to quantum level. Singularities in homogeneous classical GR are resolved: Quantum space-times can be vastly larger than what GR had us believe. Ramifications on some of the most interesting, fundamental issues.

• In $k = 0$ (and $k = 1$) FRW models with or without Λ, complete control on the physical sector of the theory. LQC evolution deterministic across the classical big bang and big crunch for all quantum states. Singularities are resolved because of unforeseen quantum geometry effects at the Planck scale. (Bianchi I: Every time a curvature invariant reaches Planck scale, quantum geometry effects intervene and dilute it: Wilson-Ewing's Talk)

• Challenge to background independent theories: Detailed recovery of classical GR at low curvatures/densities? Met in homogeneous models. As we saw from Berger & Henderson's talks, the BKL conjecture suggests a singularity resolution theorem: all space-like singularities of GR may be resolved by quantum geometry effects.

II. FAQs and Conceptual Clarifications

II.1 Why are the predictions of the WDW theory so different from those of LQC?

• Since a well-defined kinematic framework did not exist in the full WDW theory, in quantum cosmology standard quantum mechanics was used. Thus:
Kinematical Quantum States: $\Psi(a, \phi)$; $\hat{a}\Psi(a,\phi) = a\Psi(a,\phi)$ etc.
Quantum evolution governed by the Wheeler-DeWitt differential equation:

$$\ell_{\rm Pl}^4 \, \frac{\partial^2}{\partial a^2}(f(a)\Psi(a,\phi)) = {\rm const}\, G\, \hat{H}_\phi\, \Psi(a,\phi)$$

Without additional assumptions, singularity is not resolved.

• In LQC, situation is very different due to the Quantum Riemannian Geometry. How is this possible? In QM we have the von Neumann's uniqueness theorem!

- Unlike in the WDW approach, quantum kinematics in LQG is rigorously developed. The powerful uniqueness theorem! (Lewandowski, Okolow, Sahlmann, Thiemann) It provides the arena to formulate quantum Einstein equations. In LQC we could mimic this framework step by step. $\mathcal{H}_{\rm gr} = L^2(\bar{\mathbb{R}}_{\rm Bohr}, d\mu_{\rm Bohr}) \neq L^2(\mathbb{R}, dc)$. New Quantum Mechanics! (In the WDW quantum cosmology, one did not have guidance from a full theory.)

- In LQG, holonomies well defined but not connections. Like asking $U(\lambda) = \exp i\lambda x$ well-defined but operator x need not exist.
In LQC then von-Neumann's uniqueness result bypassed.
Inequivalent representations even for mini-superspaces. New quantum mechanics! Novel features precisely in the deep Planck regime.

- The LQC kinematics cannot support the WDW dynamics. The LQC dynamics is based on quantum geometry. The WDW differential equation is replaced by a difference equation. Step size governed by the smallest eigenvalue of the area operator in LQG. Good agreement with the WDW equation at low curvatures but drastic departures in the Planck regime precisely because the WDW theory ignores quantum geometry.

II.2 How is the Hamiltonian constraint 'quantized' in LQC?

- Form of the constraint $C_{\rm H} \sim \underbrace{(\epsilon^{ij}{}_k E^a_i E^b_j/\sqrt{q})}_{\rm Thiemann} \quad \underbrace{F^k_{ab}}_{\rm holonomy}$

- Classically: $F^k_{ab} = -2\lim_{{\rm Ar}_\Box\to 0}\left({\rm Tr}(h_{\Box_{ab}} - 1)\tau^k/{\rm Ar}_\Box\right)$

Quantum Theory: Limit does not exist because there is no local operator corresponding to the connection or curvature. Different from full LQG: Diff constraint handled by gauge fixing.

- LQC View (AA, Bojowald, Lewandowski,): Quantum geometry $\Rightarrow$ should not shrink the loop to zero but only till the area enclosed ${\rm Ar}\Box$ w.r.t. the fiducial metric equals the lowest eigenvalue $\Delta = 2\sqrt{3}\pi\gamma\ell_{\rm Pl}^2$ of the area operator. So, the fundamental operator has Planck scale non-locality; Familiar local expression emerges only in the classical limit. (μ_o-Scheme)

- Singularity resolved. But the resulting quantum Hamiltonian constraint had a serious limitation: Predicted deviations from the classical theory even in certain 'tame' situations. (More later). Physically motivated, improved constraint remedies this drawback while retaining all desirable features.

• New idea (AA, Pawlowski, Singh): Do this with Physical area of □ (which is state dependent). The resulting operator mimics certain features of the full theory. Idea subtle to implement but important physical consequences: Overcomes problems with the older LQC dynamics. ($\bar{\mu}$-Scheme). (more later)

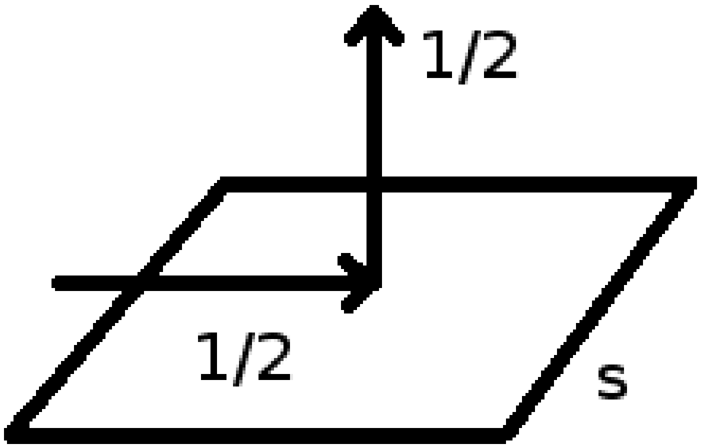
• Hamiltonian constraint: Use a representation in which geometry (i.e. $\hat{V} \sim \hat{a}^3$) and matter field (i.e., $\hat{\phi}$) are diagonal : $\Psi(v, \phi)$

Then the Wheeler DeWitt equation is replaced by a difference equation:

$$C^+(v)\,\Psi(v+4,\,\phi) + C^o(v)\,\Psi(v,\phi) + C^-(v)\Psi(v-4,\,\phi) \;=\; \hat{H}_\phi \Psi(v,\phi)$$

Fundamentally, a constraint equation. Selects physical states. However, this equation also dictates quantum dynamics.

• The 'lattice' has uniform spacing in $v \sim a^3$ (not p or μ which $\sim a^2$). Dynamics cannot be supported by a Vehlino type quantum kinematics.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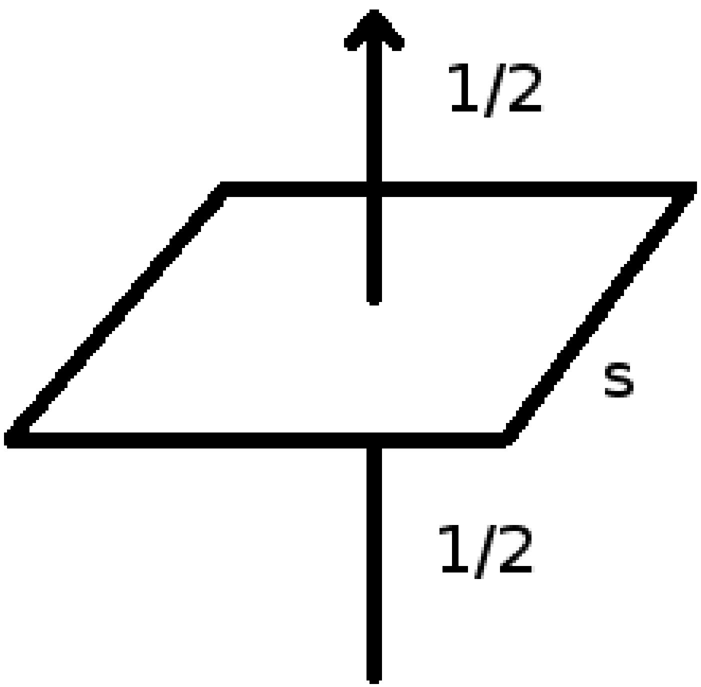
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II.3 How do you extract dynamics/physics from the ‘frozen formalism’?

To extract physics, we need to:

- Solutions to the constraint: Physical states. Introduce a physical inner product and suitable Dirac observables.

- If possible, isolate ‘time’ to give meaning to ‘evolution’. Not essential but very convenient.

- Construct states which represent the actual universe at late time. ‘Evolve back’ towards the big bang.

- Is the classical singularity ‘resolved’? In what sense? (Brunnemann and Thiemann) ‘Wave function vanishes at the singularity’ not enough; Physical inner product may be non-local. Need to analyze the behavior of the Dirac observables. Do observables which diverge at the big bang remain bounded on *physical* states?

- The quantum Hamiltonian constraint takes the form:

$$\partial_\phi^2 \Psi(v,\phi) = -\Theta\, \Psi(v,\phi) \qquad (\star)$$

where Θ is a positive, self-adjoint difference operator independent of ϕ :
$\Theta\, \Psi(v,\phi) = \; C^+(v)\, \Psi(v+4,\, \phi) + C^o(v)\, \Psi(v,\phi) + C^-(v)\, \Psi(v-4,\, \phi).$
Suggests ϕ could be used as the 'evolution parameter' or emergent time also in the quantum theory. Relational dynamics.

- Physical states: solutions to ($\star$). Observables: $\hat{p}_\phi$ and $\hat{V}|_{\phi=\phi_o}$. Inner product: Makes these self-adjoint or, equivalently, use the more general procedure of group averaging. Semi-classical states: Generalized coherent states.

- Physical states:

$$\Psi(v,\phi) \text{ satisfying } \quad -i\hbar\partial_\phi \Psi(v,\phi) = \sqrt{\Theta}\, \Psi(v,\phi)$$

Analogy with KG equation in a static space-time.

Dirac observables:

$$\hat{p}_{(\phi)}\, \Psi(v,\phi) = -i\hbar\partial_\phi\, \Psi(v,\phi) \equiv \sqrt{\Theta}\Psi(v,\phi)$$

$$\hat{V}|_\phi\, \Psi(v,\phi) = e^{i\sqrt{\Theta}(\phi-\phi_o)}\, |v|\Psi(v,\phi_o).$$ Similarly $\hat{\rho}|_\phi$.

II.4 What are the differences between the older, μ_o evolution of and the $\bar{\mu}$ framework in these models?

Differences are very significant, with interesting lessons for full LQG.

- In the k=0 model on $\mathbb{R}^3$, scale factor a refers to a fiducial metric: $q_{ab} = a^2(t)\, q^o_{ab}$. If $q^o_{ab} \to \alpha^2\, q^o_{ab}$, $a \to \alpha^{-1} a$. Physics should not depend on q^o_{ab} or the value of $a(t)$. (So, claims such as quantum effects are important for $a < a^\star$ in the older literature on homogeneous models (based on the spectrum of $1/V$ operator) are physically unsound.).

- Further, in this case *every* quantization requires an additional structure: An elementary Cell $\mathcal{C}$. We absorb factors of the volume V_o of $\mathcal{C}$ w.r.t. q^o_{ab} in the definition of canonical variables c, p so that the symplectic structure is independent of the q^o_{ab} choice. So, the classical Hamiltonian theory depends only $\mathcal{C}$ and not on q^o_{ab}. Same is true of quantum kinematics. Thus, e.g., $p^{3/2}$ is the physical volume of $\mathcal{C}$.

- i) In μ_o quantization, the Hamiltonian constraint operator depends on q_o^{ab} again. In the $\bar{\mu}$ quantization, it does not.

• ii) For each choice of $\mathcal{C}$ we get a quantum theory. In the μ_o evolution, the density at the bounce point goes as: $\rho_b \propto 1/p_\phi$. So, a Gaussian peaked at a classical phase space point can bounce with $\rho_b =$ density of water! Major departures from the classical theory also away from the bounce: in presence of a cosmological constant, large deviations occur when $\Lambda a^2 \geq 1$ although the space-time curvature is low. In $\bar{\mu}$ evolution, $\rho_b \approx 0.41\rho_{\rm pl}$ always. No departures from GR at low curvatures.

• iii) *Physical results* should be independent of the choice of $\mathcal{C}$. In $\bar{\mu}$ evolution they are. Not in the μ_o scheme. Ex: Given a classical solution $(a(t), \phi(t))$ when do quantum effects become important? Answer in the μ_o scheme depends on the choice of the cell! Answer not 'gauge invariant'. In the $\bar{\mu}$ scheme it is.

• Lessons:
a) LQC: Although it seems natural at first, detailed considerations show that the μ_o quantization of the Hamiltonian constraint is physically incorrect;
b) LQG: Whether a quantization of the Hamiltonian constraint has a 'good infra-red behavior' is likely to be very subtle. (e.g., Sakellariadou's talk)

II.5 Is there an analytical way of understanding why in numerical simulations the bounce occurs always at $\rho \approx 0.41\rho_{\rm Pl}$?

- Yes: Normally in LQC one starts by setting $N = 1$, i.e., proper time gauge and goes to ϕ as the evolution parameter only in quantum theory. If we start with ϕ (i.e., $N = a^3$), constraint simplifies. Fully analytical treatment possible (was called sLQC in the literature).

- Then matter density operator $\hat\rho = \frac{1}{2}\,(\hat V_\phi)^{-1}\,\hat p^2_{(\phi)}\,(\hat V_\phi)^{-1}$ has an absolute upper bound on the physical Hilbert space (Corichi, Singh, AA) :

$$\rho_{\rm sup} = \sqrt{3}/16\pi^2\gamma^3 G^2\hbar \approx 0.41\rho_{\rm Pl}!$$

- Furthermore, for quantum states which are Gaussians at a late time,

$$\rho_{\rm bounce} = \rho_{\rm sup}\,\left[1 - O\Big(\frac{G\hbar^2}{p^2_{(\phi)} + (\Delta p_{(\phi)})^2}\Big)\right]$$

II.6 Is the bounce restricted only to semi-classical states?

- No. In sLQC, for any normalized state Ψ in the physical Hilbert space we have:

$$(\Psi,\, \hat{V}_\phi \Psi)_{\rm Phy} = V_+ e^{\sqrt{12\pi G}\phi} + V_- e^{-\sqrt{12\pi G}\phi}$$

where $V_\pm$ are determined by the 'initial data' $\Psi(v, \phi_o)$ at any ϕ_o. So:

$$V_{\rm min} = \sqrt{(V_- V_+)} \quad \text{and} \quad \phi_{\rm bounce} = \frac{1}{2\sqrt{12\pi G}}\,(\ln V_- - \ln V_+)$$

II.7 Is there a precise relation between the WDW theory and LQC?

Question analyzed in detail for the exactly soluble k=0 model (Corichi, Singh, AA).

Start with the same physical state at $\phi = \phi_o$ and evolve using sLQC or WDW theory. Then:

- Certain predictions of LQC approach those of the WDW theory as the area gap λ goes to zero:
  Given a semi-infinite 'time' interval $\Delta\phi$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $\forall\lambda < \delta$, 'physical predictions of the two theories are within ϵ of each other.'
- However, approximation is *not* uniform. The WDW theory is *not* the limit of sLQC:
  Given $N > 0$ however large, there exists a ϕ such that
  $\langle\hat{V}_\phi\rangle_{\rm sLQC} - \langle\hat{V}_\phi\rangle_{\rm WDW} > N$.
  LQC is *fundamentally* discrete.

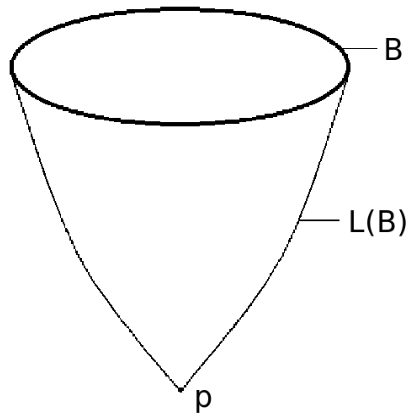
II.8 What happens to the Bousso entropy bound in LQC?

- Conjecture (Simplest Version): The matter entropy flux across $\mathcal{L}(\mathcal{B})$ is bounded by

$$S := \int_{\mathcal{L}(\mathcal{B})} S^a dA_a \leq A_{\mathcal{B}}/4{\ell_{\rm pl}}^2 .$$

- Curious features:

i) Requires a notion of entropy current;
ii) Refers to quantum gravity;
iii) Requires a classical geometry.
Consequently, quite difficult to test in practice!



- In classical GR:

If we consider k=0 FRW models filled with radiation,

$$\frac{S}{A_{\mathcal{B}}} = \frac{{\ell_{\rm pl}}^2}{6}\,\left(\frac{2}{45\pi}\right)^{1/4}\,\frac{\sqrt{\ell_{\rm pl}}}{\sqrt{\tau_f}}\,\left(1 - \sqrt{\frac{\tau_i}{\tau_f}}\right)$$

For round $\mathcal{B}$, the bound holds if $\tau_f > 0.1\ell_{\rm pl}$ but arbitrarily large violations near the singularity.

- LQC provides an ideal arena:

i) Singularity is resolved by quantum gravity;
ii) The wave function is sharply peaked about a mean metric, a smooth field (although coefficients involve $\hbar$).

- Answer: $\frac{S}{A_{\mathcal{B}}} < 0.244/{\ell_{\rm pl}}^2$ (AA, Wilson-Ewing)

The bound is satisfied in LQC!

II.9 LQC is just a toy model. Why should one be interested in it?

Physically important examples can be powerful.

- Full QED versus Dirac's hydrogen atom.
- Singularity Theorems versus first discoveries in simple models.
- BKL behavior: homogeneous Bianchi models.
- Since inhomogeneities can be treated as perturbations classically, one could do QFT for them on a quantum geometry provided by LQC of homogeneous modes (Lewandowski's talk); or, through Fock quantization in a 'hybrid' approach (Mena's talk).

Does *not* imply that behavior found in examples is necessarily generic. Rather, they can reveal important aspects of the full theory. Can work one's way up by considering more and more complicated cases. At each step, physical checks well beyond formal mathematics. Can have strong lessons for the full theory. For example, LQC has taught us that loopy techniques do capture sectors with good semi-classical behavior but only if the Hamiltonian constraint is quantized in a certain way.

Summary

- Key differences between LQC and WDW theory arise because of quantum geometry. In the limit that the area gap shrinks to zero, LQC reduces to the WDW theory.

- LQC provides explicit examples where the complete physical sector can be constructed. Provides a clear sense in which the singularity is resolved. Lessons on how to extract dynamics from the frozen formalism.

- Already in the simple FRW models, quantization of the Hamiltonian constraint turned out to be far more subtle than one could have imagined. The μ_o scheme looked natural at first. But turned out not to be viable once the physical sector was constructed and analyzed in detail. Important lessons for full LQG.

- The Bousso entropy bound is violated in classical GR near the big bang singularity. But respected in LQC. Confluence of divergent ideas. But the analysis also shows that the entropy bound does not have to be a fundamental building block of quantum gravity but may emerge in appropriate conditions from quantum geometry.

Supplement: LQC Kinematics

- In LQG the canonically conjugate variables are:

A_a^i, SU(2) connections and, E_i^a, orthonormal triads.

Spatial homogeneity and isotropy implies

$$\star \qquad A_a = c\,\underbrace{\mathring{\omega}_a^i \sigma_i}_{\text{fixed}}, \qquad E^a = p\,\underbrace{\mathring{e}_i^a \sigma^i}_{\text{fixed}}$$

–c: $\sim \dot{a}$

–holonomy: $h_e(c) = \cos\mu c\ \mathbf{1} + \sin\mu c\ \dot{e}^a \mathring{\omega}_a^i \sigma_i$

(Almost periodic in c)

– $|p| = a^2$.

– $p \to -p$ changes only the orientation of the triad.

Large gauge transformation; leaves physics invariant.

$\star$ Canonically conjugate pairs:

c, p for gravity $\qquad$ ϕ, p_ϕ for matter

- Loop quantum cosmology:

Key strategy:

Do not naively set $\mathcal{H} = L^2(R, \mathrm{dc})$ and $\hat{c}\Psi(c) = c\Psi(c)$; $\hat{p}\Psi(c) = -i\hbar \frac{\mathrm{d}\Psi}{\mathrm{dc}}$.

Rather, Follow full theory.

New Quantum Mechanics

$\star$ States: depend on c only through holonomies
$\Rightarrow$ Almost periodic functions of c, $\sim \exp i\mu c$ where $\mu \in R$.
$\star$ Operators: holonomies $\hat{h}_e(c)$ act by multiplication
Momentum fluxes $\sim \hat{p} = -i\hbar \frac{\mathrm{d}}{\mathrm{dc}}$

Full theory suggests: No operator $\hat{c}$ corresponding to c; i.e., $\hat{h}_e(c)$ not weakly continuous in $e \sim \mu \Rightarrow$ von Neumann's uniqueness theorem by-passed. New Quantum Mechanics possible (Bojowald, Lewandowski, AA) .

- Differences from standard quantum mechanics:

$\star$ States: Built from holonomies: $\Psi(c) = \sum \alpha_j \, e^{i\mu_j c}$

where $\mu_j \in \mathbf{R};\ \alpha_j \in \mathbf{C};\ \sum_j |\alpha_j|^2 < \infty$

$$\mathcal{H}_{\rm grav} = L^2(\bar{\mathbf{R}}_{\rm Bohr}, d\mu_o) \quad \neq L^2(\mathbf{R}, dc)$$

$\star$ operators: $\hat{h}$, $\hat{p}$ well-defined. But No $\hat{c}$.
Spectrum of $\hat{p}$ is the real line with *discrete* topology; eigenstates $e^{i\mu c}$ normalizable.

- Structure mimics that of the full theory. The new kinematics does not support WDW dynamics.