Non-Commutative/Non-Associative Geometry and Non-geometric String Backgrounds

DIETER LÜST (LMU, MPI)

Workshop on Non-Associativity in Physics and Related Mathematical Structures, PennState, 1st May 2014
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Outline:

I) Introduction

II) Non-commutative/non-associative closed string geometry and non-geometric string backgrounds

III) Some Remarks on Double Field Theory

Talk by Ralph Blumenhagen
I) Introduction

Two complementary approaches to quantum gravity:
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- Canonical quantum gravity (LQG, CDT) for point-like fields:

  Discrete (non-commutative) fuzzy space-time:
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Two complementary approaches to quantum gravity:

- Canonical quantum gravity (LQG, CDT) for point-like fields:
  
  Discrete (non-commutative) fuzzy space-time:

- String theory (finitely extended objects):
  
  Smooth geometry (resolution of singularities)
I) Introduction

Two complementary approaches to quantum gravity:

- Canonical quantum gravity (LQG, CDT) for point-like fields:
  
  Discrete (non-commutative)
  fuzzy space-time:

- String theory (finitely extended objects):

  Smooth geometry
  (resolution of singularities)

What is the relation between these two approaches?
As we will see, non-geometric string backgrounds and T-duality will provide a very interesting and new classical relation between fuzzy space and finite extension of the string.
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⇒ Non-commutative and non-associative closed string geometry:

A. Bakas, D. Lüst, arXiv:1309.3172
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⇒ Non-commutative and non-associative closed string geometry

"Fundamental" closed string non-associativity in WZW-model with H-flux

Closed string non-commutativity in "tori" with non-geometric fluxes and T-duality, "derived" non-associativity follows as violation of Jacobi identity

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\[ \Rightarrow \text{Non-commutativity in "tori" with non-geometric fluxes and T-duality, non-associativity follows as violation of Jacobi identity} \]

Non-associativity in CFT’s with geometric and T-dual non-geometric fluxes

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⇒ Non-commutativity in closed string geometry

Non-associativity in CFT's with geometric and T-dual non-geometric fluxes

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A. Bakas, D. Lüst, arXiv:1309.3172

⇒ New classical uncertainty relations - minimal volume due to finite string length!
Non-associativity in physics:

- Jordan & Malcev algebras, octonions

- Nambu dynamics
  Y. Nambu (1973); D. Minic, H. Tze (2002); M. Axenides, E. Floratos (2008)

- Magnetic monopoles
  R. Jackiw (1985); M. Günaydin, B. Zumino (1985)

- Closed string field theory

- T-duality and principle torus bundles

- D-branes in curved backgrounds
  L. Cornalba, R. Schiappa (2001)

- Multiple M2-branes and 3-algebras
  J. Bagger, N. Lambert (2007)
II) Non-geometric backgrounds and non-commutative & non-associative geometry
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Closed string background fields: $G_{i j}$, $B_{i j}$, $\Phi$
II) Non-geometric backgrounds and non-commutative & non-associative geometry

Closed string background fields: \( G_{ij}, B_{ij}, \Phi \)

Generalized metric: \( \mathcal{H}_{MN} = \begin{pmatrix} G^{ij} & -G^{ik}B_{kj} \\ B_{ik}G^{kj} & G_{ij} - B_{ik}G^{kl}B_{lj} \end{pmatrix} \)
II) Non-geometric backgrounds and non-commutative & non-associative geometry

Closed string background fields: \( G_{ij}, B_{ij}, \Phi \)

Generalized metric:
\[
\mathcal{H}_{MN} = \begin{pmatrix}
G^{ij} & -G^{ik}B_{kj} \\
B_{ik}G^{kj} & G_{ij} - B_{ik}G^{kl}B_{lj}
\end{pmatrix}
\]

T-duality - \( O(D,D) \) transformations:
\[
\mathcal{H}_{MN} \rightarrow \Lambda^P_M \mathcal{H}_{PQ} \Lambda^Q_N, \quad \Lambda \in O(D,D)
\]

They contain:
\( B_{ij} \rightarrow B_{ij} + 2\pi \Lambda_{ij}, \quad R \rightarrow L_s^2/R \)
II) Non-geometric backgrounds and non-commutative & non-associative geometry

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String length (not \( hbar! \))
II) Non-geometric backgrounds and non-commutative & non-associative geometry

Closed string background fields: \( G_{ij}, B_{ij}, \Phi \)

Generalized metric: \( \mathcal{H}_{MN} = \begin{pmatrix} G^{ij} & -G^{ik} B_{kj} \\ B_{ik} G^{kj} & G_{ij} - B_{ik} G^{kl} B_{lj} \end{pmatrix} \)

T-duality - \( O(D,D) \) transform:

\( \mathcal{H}_{MN} \rightarrow \Lambda_{MN} \) \( \Lambda \in O(D,D) \)

They contain:

\( B_{ij} \rightarrow B_{ij} + 2\pi \lambda_{ij}, \quad R \rightarrow \frac{L_s^2}{R} \)

Doubling of closed string coordinates and momenta:

- Coordinates: \( O(D,D) \) vector

\( X^M = (\tilde{X}_i, X^i) \)

- Momenta: \( O(D,D) \) vector

\( p^M = (\tilde{p}^i, p_i) \)

(Here \( D=3 \))
Non-geometric backgrounds are generic within the landscape of string „compactifications“. Several potentially interesting applications in string phenomenology and cosmology.

- They are only consistent in string theory.
- Make use of string symmetries, T-duality \( \Rightarrow \) T-folds,
- Left-right asymmetric spaces \( \Rightarrow \) Asymmetric orbifolds
  

- They can be potentially used for the construction of de Sitter vacua

- T-duality: classical bounce (pre-big bang) models
  
  (Brandenberger, Vafa, 1989; Meissner, Veneziano, 1991; Gasperini, Veneziano, 1993)
Non-geometric Q-fluxes: spaces that are locally still Riemannian manifolds but not anymore globally.

(Hellerman, McGreevy, Williams (2002); C. Hull (2004); Shelton, Taylor, Wecht, 2005; Dabholkar, Hull, 2005)

Transition functions between two coordinate patches are given in terms of $O(D,D)$ T-duality transformations:

$$\text{Diff}(M_D) \rightarrow O(D, D)$$

Q-space will become non-commutative: $[X^i, X^j] \neq 0$
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Transition functions between two coordinate patches are given in terms of $O(D,D)$ T-duality transformations:

\[
\text{Diff}(M_D) \rightarrow O(D, D)
\]

C. Hull (2004)

**Q-space will become non-commutative:**

\[
[X^i, X^j] \neq 0
\]

- **Non-geometric R-fluxes:** spaces that are even locally not anymore manifolds.

**R-space will become non-associative:**

\[
[X^i, X^j, X^k] := [[X^i, X^j], X^k] + \text{cycl. perm.} = (X^i \cdot X^j) \cdot X^k - X^i \cdot (X^j \cdot X^k) + \cdots \neq 0
\]
Example: Three-dimensional flux backgrounds:

Fibrations: 2-dim. torus that varies over a circle:

\[ T^{2}_{X^{1}, X^{2}} \hookrightarrow M^{3} \hookrightarrow S^{1}_{X^{3}} \]

The fibration is specified by its monodromy properties.

Metric, B-field of \( T^{2} \):

\[ \mathcal{H}_{MN}(X^{3}) \]

\( \text{O}(2,2) \) monodromy:

\[ \mathcal{H}_{MN}(X^{3} + 2\pi) = \Lambda_{O(2,2)} \mathcal{H}_{PQ}(X^{3}) \Lambda_{O(2,2)}^{-1} \]
Example: Three-dimensional flux backgrounds:

Fibrations: 2-dim. torus that varies over a circle:

\[ T_{X^1,X^2}^2 \leftrightarrow M^3 \leftrightarrow S^1_{X^3} \]

The fibration is specified by its monodromy properties.

O(2,2) monodromy: \( \mathcal{H}_{MN}(X^3 + 2\pi) = \Lambda_{O(2,2)} \mathcal{H}_{PQ}(X^3) \Lambda_{O(2,2)}^{-1} \)

Complex structure \( \tau \) of \( T^2 \): \( \tau(X^3 + 2\pi) = \frac{a\tau(X^3) + b}{c\tau(X^3) + d} \)

Kähler parameter \( \rho \) of \( T^2 \): \( \rho(X^3 + 2\pi) = \frac{a'\rho(X^3) + b'}{c'\rho(X^3) + d'} \)
Torus
Torus with non-constant B-field (H-flux), B-field is patched together by a B-field (gauge) transformation: \[ B \rightarrow B + 2\pi H \]
Non geometric torus, metric is patched together by a T-duality transformation: \[ G_{ij} \rightarrow G^{ij} \]
Non geometric torus, metric is patched together by a T-duality transformation:

$$G_{ij} \rightarrow G^{ij}$$
3-dimensional fibration: $\tau(X^3 + 2\pi) = -\frac{1}{\tau(X^3)}$

Twisted torus with f-flux
3-dimensional fibration: \( \rho(X^3 + 2\pi) = -\frac{1}{\rho(X^3)} \)

Non-geometric space with Q-flux

\( T^2_{X^1 X^2} \)

\( S^1 \)

\( S^1_{X^3} \)
(i) (Non-)geometric backgrounds with parabolic monodromy and constant 3-form fluxes:

Chain of four T-dual spaces:

(a) Geometric space: 3-dimensional torus with $H$-flux

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{12} = H X_H^3, \quad H_{123} = \partial_3 B_{12} = H$$

$$X_H^3 \rightarrow X_H^3 + 2\pi R_3 \quad \Rightarrow \quad g_{O(2,2)} : \rho(X_H^3 + 2\pi R_3) = \rho(X_H^3) + 2\pi H R_3$$
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\[ \rho(X_H^3) = i R_1 R_2 - H X_H^3 \]

\[ X_H^3 \rightarrow X_H^3 + 2\pi R_3 \quad \Rightarrow \quad g_{O(2,2)} : \rho(X_H^3 + 2\pi R_3) = \rho(X_H^3) + 2\pi HR_3 \]

T-duality in \( X^1 \):

(b) Geometric spaces: twisted 3-torus with \( f \) - flux \((f \equiv H)\)

\[ G_{ij} = \begin{pmatrix} \frac{1}{R_1^2} & -\frac{f X_f^3}{R_1^2} & 0 \\ -\frac{f X_f^3}{R_1^2} & R_2^2 + \left(\frac{f X_f^3}{R_1}\right)^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{ij} = 0 \]

\[ \tau(X_f^3) = i R_1 R_2 - f X_f^3 \]

\[ X_f^3 \rightarrow X_f^3 + 2\pi R_3 \quad \Rightarrow \quad g_{O(2,2)} : \tau(X_f^3 + 2\pi R_3) = \tau(X_f^3) + 2\pi f R_3 \]
T-duality in $X^2$:

(c) Non-geometric space: T-fold with Q-flux ($Q \equiv f \equiv H$)

$$G_{ij} = \begin{pmatrix} \frac{F}{R_1^2} & 0 & 0 \\ 0 & \frac{F}{R_2^2} & 0 \\ 0 & 0 & \frac{F}{R_3^2} \end{pmatrix}, \quad B_{ij} = F \begin{pmatrix} 0 & -\frac{QX_Q^3}{R_1^2 R_2^2} & 0 \\ \frac{QX_Q^3}{R_1^2 R_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \left(1 + \left(\frac{QX_Q^3}{R_1 R_2}\right)^2\right)^{-1}$$

$$\rho(X_Q^3) = \frac{1}{QX_Q^3 - iR_1 R_2} \quad \Rightarrow \quad g_{O(2,2)} : \rho(X_Q^3 + 2\pi R_3) = \frac{\rho(X_Q^3)}{1 + 2\pi R_3 Q \rho(X_Q^3)}$$

This does not correspond to a standard diffeomorphism but to a T-duality transformation.
T-duality in $X^2$:

(c) Non-geometric space: T-fold with Q-flux \((Q \equiv f \equiv H)\)

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G_{ij} = \begin{pmatrix} \frac{F}{R_1^2} & 0 & 0 \\ 0 & \frac{F}{R_2^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{ij} = F \begin{pmatrix} 0 & \frac{-QX_Q^3}{R_1^2R_2^2} & 0 \\ \frac{QX_Q^3}{R_1^2R_2^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \left(1 + \left(\frac{QX_Q^3}{R_1R_2}\right)^2\right)^{-1}
\]

\[
\rho(X_Q^3) = \frac{1}{QX_Q^3 - iR_1R_2} \Rightarrow g_{O(2,2)} : \rho(X_Q^3 + 2\pi R_3) = \frac{\rho(X_Q^3)}{1 + 2\pi R_3Q \rho(X_Q^3)}
\]

This does not correspond to a standard diffeomorphism but to a T-duality transformation.

T-duality in $X^3$:

(d) Non-geometric space with R-flux

Now the Buscher rules for T-duality cannot be applied.

There exist no locally defined metric and B-field.
Summary:

Flat torus with H-flux $T x_1 \leftrightarrow$ Twisted torus with f-flux $T x_2 \leftrightarrow$ Non-geometric space with Q-flux $T x_3 \leftrightarrow$ Non-geometric space with R-flux
Summary:

Flat torus with H-flux $T_{x_1}$ Twisted torus with f-flux $T_{x_2}$ Non-geometric space with Q-flux $T_{x_3}$ Non-geometric space with R-flux

$[X^i_{H,f}, X^j_{H,f}] = 0$
Summary:

- Flat torus with H-flux
- Twisted torus with f-flux
- Non-geometric space with Q-flux
- Non-geometric space with R-flux

\[ [X^i_{H,f}, X^j_{H,f}] = 0 \]

\[ [X^1_Q, X^2_Q] \sim Q \tilde{p}^3 \]
Summary:

- Flat torus with H-flux
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\[ [X^i_{H,f}, X^j_{H,f}] = 0 \]

\[ [X^1_Q, X^2_Q] \simeq Q \hat{p}^3 \]

\[ [[[X^1_R, X^2_R], X^3_R] \simeq R \]
Summary:

\[
\begin{align*}
[X^i_{H,f}, X^j_{H,f}] &= 0 \\
[X^1_Q, X^2_Q] &\simeq Q \tilde{p}^3 \\
[[X^1_R, X^2_R], X^3_R] &\simeq R
\end{align*}
\]

They can be computed by

- standard world-sheet quantization of the closed string

- CFT & canonical T-duality
  I. Bakas, D.L. to appear soon
Q-flux:

\[ [X^1_Q(\tau, \sigma), X^2_Q(\tau, \sigma')] = \]

\[ -\frac{i}{2} Q \hat{p}^3 \left( \sum_{n \neq 0} \frac{1}{n^2} e^{-in(\sigma' - \sigma)} - (\sigma' - \sigma) \sum_{n \neq 0} \frac{1}{n} e^{-in(\sigma' - \sigma)} + \frac{i}{2}(\sigma' - \sigma)^2 \right) \]
Q-flux:

\[
\left[ X_Q^1(\tau, \sigma), X_Q^2(\tau, \sigma') \right] = -\frac{i}{2} Q \hat{p}^3 \left( \sum_{n \neq 0} \frac{1}{n} e^{-i n (\sigma' - \sigma)} - (\sigma' - \sigma) \sum_{n \neq 0} \frac{1}{n} e^{-i n (\sigma' - \sigma)} + \frac{i}{2} (\sigma' - \sigma)^2 \right)
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\[
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\[
-\frac{i}{2} Q \tilde{p}^3 \left( \sum_{n \neq 0} \frac{1}{n^2} e^{-i n (\sigma' - \sigma)} - (\sigma' - \sigma) \sum_{n \neq 0} \frac{1}{n} e^{-i n (\sigma' - \sigma)} + \frac{i}{2} (\sigma' - \sigma)^2 \right)
\]

\[\sigma \rightarrow \sigma':\]

\[
\left[ X_Q^1(\tau, \sigma), X_Q^2(\tau, \sigma) \right] = -i \frac{\pi^2}{6} Q \tilde{p}^3
\]

The non-commutativity of the torus (fibre) coordinates is determined by the winding in the circle (base) direction.
Corresponding uncertainty relation:

\[ (\Delta X_Q^1)^2 (\Delta X_Q^2)^2 \geq L_s^6 Q^2 \langle \tilde{p}^3 \rangle^2 \]

The spatial uncertainty in the \( X_1, X_2 \) - directions grows with the dual momentum in the third direction: non-local strings with winding in third direction.
R-flux background: T-duality in $x^3$-direction $\Rightarrow$ R-flux

\[ \tilde{p}^3 \longleftrightarrow p_3, \quad \tilde{X}_{Q,3} \equiv X_R^3 \]

$\Rightarrow$ For the case of non-geometric R-fluxes one gets:

\[
[X_R^1, X_R^2] = -i \frac{\pi^2}{6} R p_3
\]

$R \equiv Q$

Use $[X_R^3, p_3] = i \quad \Rightarrow$ Non-associative algebra:

\[
[[X_R^1(\tau, \sigma), X_R^2(\tau, \sigma)], X_R^3(\tau, \sigma)] + \text{perm.} = \frac{\pi^2}{6} R
\]
**R-flux background:** T-duality in $x^3$-direction $\Rightarrow$ R-flux

$$\tilde{p}^3 \longleftrightarrow p_3 \text{ , } \tilde{X}_{Q,3} \equiv X^3_R$$

$\Rightarrow$ For the case of non-geometric R-fluxes one gets:

$$[X^1_R, X^2_R] = -i\frac{\pi^2}{6} R p_3 \text{ , } R \equiv Q$$

Use $[X^3_R, p_3] = i \Rightarrow$ Non-associative algebra:

$$[[X^1_R(\tau, \sigma), X^2_R(\tau, \sigma)], X^3_R(\tau, \sigma)] + \text{perm.} = \frac{\pi^2}{6} R$$

**Corresponding classical „uncertainty relations“:**

$$(\Delta X^1_R)^2 (\Delta X^2_R)^2 \geq L_s^6 R^2 \langle p^3 \rangle^2$$

Volume: $$(\Delta X^1_R)^2 (\Delta X^2_R)^2 (\Delta X^3_R)^2 \geq L_s^6 R^2$$

(see also: D. Mylonas, P. Schupp, R. Szabo, arXiv:1312.1621)
The algebra of commutation relation looks different in each of the four duality frames.

Non-vanishing commutators and 3-brackets:

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<td><strong>f-flux</strong></td>
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However: R-flux & winding coordinates: $[\tilde{x}^i, \tilde{x}^j, \tilde{x}^k] = 0$
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However: $R$-flux & winding coordinates: $[\tilde{x}^i, \tilde{x}^j, \tilde{x}^k] = 0$

T-duality is mapping these commutators and 3-brackets onto each other:
(ii) (Non-)geometric backgrounds with \textit{elliptic monodromy} and non-geometric fluxes.

They can be described in terms of twisted tori and (a)symmetric freely acting orbifolds.

D. Lüst, JHEP 1012 (2011) 063, arXiv:1010.1361,
C. Condeescu, I. Florakis, C. Kounnas, D.Lüst, arXiv:1307.0999

\begin{itemize}
\item In general not T-dual to a geometric space!
\item (Only consistent in string theory (respectively in DFT).)
\end{itemize}

The fibre torus depends on the third coordinate in a more complicate way.
The corresponding commutators can be explicitly derived in CFT.

More complicate, non-linear commutation relations:

R-frame:

\[
[x^1, x^2] = [\tilde{x}^1, x^2] = [x^1, \tilde{x}^2] = [\tilde{x}^1, \tilde{x}^2] = i\Theta(p_3)
\]

\[
\Theta(p_3) = \frac{\pi}{2} \cot(\pi p_3 R)
\]

This algebra cannot be T-dualized to a commutative algebra!

The string always moves on a non-commutative/non-associative fuzzy space:
Mathematical framework to describe non-geometric string backgrounds and the non-associative algebras:


I. Bakas, D. Lüst, arXiv:1309.3172;

⇒ 3-Cocycles, 2-cochains and star-products

- Group theory cohomology - Hochschild; Stasheff; Cartan, Eilenberg, ...
Mathematical framework to describe non-geometric string backgrounds and the non-associative algebras:


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Open string non-commutativity:

Constant Poisson structure:

$[x_i, x_j] = \theta_{ij}$
Mathematical framework to describe non-geometric string backgrounds and the non-associative algebras:


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⇒ 3-Cocycles, 2-cochains and star-products

- Group theory cohomology - Hochschild; Stasheff; Cartan, Eilenberg, ...

Open string non-commutativity:

Constant Poisson structure:

\[
\left[ x_i, x_j \right] = \theta_{i j}
\]

Moyal-Weyl star-product:

\[
(f_1 \star f_2)(\vec{x}) = e^{i\theta_{ij}} \partial_i x^1 \partial_j x^2 f_1(\vec{x}_1) f_2(\vec{x}_2)|_{\vec{x}}
\]

2-cyclicity:

\[
\int d^n x \ (f \star g) = \int d^n x \ (g \star f)
\]

Non-commutative gauge theories:

\[
S \simeq \int d^n x \ \text{Tr} \hat{F}_{ab} \star \hat{F}^{ab}
\]

(N. Seiberg, E. Witten (1999); J. Madore, S. Schraml, P. Schupp, J. Wess (2000); ....)
Are the similar structures for closed strings?

⇒ Tri-product.

Deformed theory of gravity? Possibly yes, but only off-shell.
Are the similar structures for closed strings?

⇒ Tri-product.

Deformed theory of gravity? Possibly yes, but only off-shell.

Recall: closed string parabolic model in R-flux frame:

Non-associative algebra:

\[ [x^i, x^j] \sim l_s^3 \hbar^{-1} R \epsilon^{ijk} p_k \]

\[ [x^i, p^j] = i\hbar \delta^{ij}, \quad [p^i, p^j] = 0 \]

\[ [x^1, x^2, x^3] := [[x^1, x^2], x^3] + \text{cycl. perm.} \sim l_s^3 R \]
3-cocycles in Lie group cohomology

Consider the following group elements (loops):

\[ U(\vec{a}, \vec{b}) = e^{i(\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{p})} \]

Now we want to consider the product of two or three group elements in order to derive the non-commutative/non-associative phases in the group products (BCH formula).
Group cohomology:

\[ U(\vec{a}_1, \vec{b}_1)U(\vec{a}_2, \vec{b}_2) = e^{-i \frac{\pi^2 R}{12} \varphi(\vec{a}_1, \vec{a}_2)} U(\vec{a}_2, \vec{b}_2)U(\vec{a}_1, \vec{b}_1). \]

Non-commutativity is determined by the following 2-cochain:

\[ \varphi_2(\vec{a}_1, \vec{a}_2) = (\vec{a}_1 \times \vec{a}_2) \cdot \vec{p} \]
Product law of three group elements becomes non-associative:

\[
\left( U(\vec{a}_1, \vec{b}_1) U(\vec{a}_2, \vec{b}_2) \right) U(\vec{a}_3, \vec{b}_3) = e^{-i \frac{R}{2} (\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3} U(\vec{a}_1, \vec{b}_1) \left( U(\vec{a}_2, \vec{b}_2) U(\vec{a}_3, \vec{b}_3) \right).
\]

Non-associativity is determined by the 3-cocycle:

\[
\varphi_3(\vec{a}_1, \vec{a}_2, \vec{a}_3) = (\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3
\]

3-cocycle: volume of tetrahedron:

\[
\text{Volume}(\vec{a}_1, \vec{a}_2, \vec{a}_3) = \frac{1}{6} |(\vec{a}_1 \times \vec{a}_2) \cdot \vec{a}_3|
\]
Derivation of the star product

The multiplication of the group elements and the use of Weyl's correspondence rule lead to star 2- and 3-products for the multiplication of functions $f(\vec{x}, \vec{p})$.

$$(f_1 \star_p f_2)(\vec{x}, \vec{p}) = e^{\frac{i}{2} \theta^{IJ}(p) \partial_I \otimes \partial_J} (f_1 \otimes f_2)|_{\vec{x}; \vec{p}}$$

6-dimensional Poisson tensor:

$$\theta^{IJ}(p) = \begin{pmatrix} R^{ijk} p_k & \delta^i_j \\ -\delta^j_i & 0 \end{pmatrix}; \quad R^{ijk} = \frac{\pi^2 R}{6} \epsilon^{ijk}$$

(The full phase space including also dual coordinates and dual momenta is 12-dimensional!)


I. Bakas, D. Lüst, arXiv:1309.3172
It leads to the following 3-product:

\[(f_1 \triangle_3 f_2 \triangle_3 f_3)(\vec{x}) = ((f_1 \ast_p f_2) \ast_p f_3)(\vec{x})\]

\[(f_1 \triangle_3 f_2 \triangle_3 f_3)(\vec{x}) = e^{iR^{ijk}} \partial_{x^1} \partial_{x^2} \partial_{x^3} f_1(\vec{x}_1) f_2(\vec{x}_2) f_3(\vec{x}_3)|_{\vec{x}}\]

This delta-product is non-associative.

It is consistent with the 3-bracket among the coordinates:

\[f_1 = x^i, f_2 = x^j, f_3 = x^k:\]

\[f_1 \triangle_3 f_2 \triangle_3 f_3 = [x^i, x^j, x^k] = \ell^4_s R^{ijk}\]

It obeys the 3-cyclicity property:

\[\int d^nx (f_1 \triangle_3 f_2) \triangle_3 f_3 = \int d^nx f_1 \triangle_3 (f_2 \triangle_3 f_3)\]
\[ \Delta_3 \text{ was already derived in CFT from the multiplication of 3 tachyon vertex operators:} \]


**Scattering of 3 momentum states in R-background:**
(corresponds to 3 winding states in H-background)

\[ V_i(z, \bar{z}) =: \exp(ip_i X^i(z, \bar{z})) : \]

\[ \langle V_{\sigma(1)} V_{\sigma(1)} V_{\sigma(1)} \rangle_R = \langle V_1 V_2 V_3 \rangle_R \times \exp(-i\eta_\sigma R^{ijk} p_{1,i} p_{2,j} p_{3,k}). \]

\[ (\eta_\sigma = 0, 1) \]

However this non-associative phase is vanishing, when going on-shell in CFT and using momentum conservation:

\[ p_1 = -(p_2 + p_3) \]

**On-shell CFT amplitudes are associative!**
IV) Double geometry - double field theory

W. Siegel (1993); C. Hull, B. Zwiebach (2009); C. Hull, O. Hohm, B. Zwiebach (2010,...)

Effective field theory description of non-geometric spaces:
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Effective field theory description of non-geometric spaces:

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(still T-dual to geometric spaces)

D. Andriot, M. Larfors, D. Lüst, P. Patalong, arXiv:1106.4015,
D. Andriot, A. Betz, arXiv:1306.4381,
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(iii) Non-geometric (H,f,Q,R)-spaces: Double field theory

O. Hohm, D. Lüst, B. Zwiebach, arXiv:1309.2977,
R. Blumenhagen, M. Fuchs, Hassler, D. Lüst, R. Sun, arXiv:1312.0719,
Double field theory:

- $O(D,D)$ invariant effective string action containing momentum and winding coordinates at the same time:

$$X^M = (\tilde{x}_m, x^m)$$
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- Covariant flux formulation of DFT.
  
  (Geissbuhler, Marques, Nunez, Penas; Aldazabal, Marques, Nunez)

Comprises all fluxes (Q,f,Q,R) into one covariant expression:

\[ \mathcal{F}_{ABC} = \mathcal{D}_{[A} E_{B}^{M} E_{C]} M, \quad \mathcal{D}^{A} = E^{A}_{M} \partial^{M}. \]
Double field theory:

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Comprises all fluxes (Q,f,Q,R) into one covariant expression:

\[ \mathcal{F}_{ABC} = \mathcal{D} [A E_B^M E_C]_M, \quad \mathcal{D}^A = E^A_M \partial^M. \]

- The DFT action is invariant under generalized, non-associative diffeomorphisms:
  \[ \xi^M = (\tilde{\lambda}_m, \lambda^m) \]
  \[ \delta \xi E^A_M = \mathcal{L}_\xi E^A_M = \xi^P \partial_P E^A_M + (\partial_M \xi^P - \partial^P \xi_M) E^A_P \]

  \[ [\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]} \]

  \[ [\xi_1, \xi_2]^M_C = \xi_1^N \partial_N \xi_2^M - \frac{1}{2} \xi_1^N \partial^M \xi_2^N - (\xi_1 \leftrightarrow \xi_2) \]
The generalized diffeomorphisms contain simultaneous coordinate and B -, $\beta$ - gauge field transformations.

Generalized diffeomorphisms act on the generalized coordinates in a non-associative way:

The Courant bracket violates the Jacobi identity.

C. Hull, B. Zwiebach, arXiv:0908.1792;
O. Hohm, B. Zwiebach, arXiv:1207.4198;
However for generalized functions $f(X)$ (e.g. the background fields) one has to require the strong constraint (string level matching condition):

\[ \partial_M \partial^M = 0, \quad \partial_M f \partial^M g = \mathcal{D}_A f \mathcal{D}^A g = 0 \]

Functions must depend only on one kind of coordinates.

The strong constraint defines a $D$-dim. hypersurface (brane) in 2D-dim. double geometry.

Then the algebra of diffeomorphisms on functions closes and becomes associative.
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Patching of coordinate charts correspond to generalized coordinate transformations of the form

\[
X'^M = X^M - \partial^M \chi,
\]

where the gauge functions \( \chi \) in general depend on \( x^m \) and \( \tilde{x}_m \).
Consider the following 3+3 dimensional backgrounds:

(i) **Parabolic** background spaces:  

\[ H_{123} \text{ or } f_{23}^1 \text{ or } Q_{3}^{12} \text{ or } R^{123} \]

These backgrounds do not satisfy \[ R^{MN} = 0 \].
Consider the following 3+3 dimensional backgrounds:

(i) **Parabolic** background spaces: Single fluxes:

\[ \begin{align*}
H_{123} \quad & \text{or} \quad f_{1}^{23} \quad & \text{or} \quad Q_{3}^{12} \quad & \text{or} \quad R^{123}
\end{align*} \]

These backgrounds do not satisfy \( R^{M N} = 0 \).

(ii) **Elliptic** background spaces: Multiple fluxes:

These backgrounds do satisfy \( R^{M N} = 0 \).

- Single elliptic geometric space: \( f_{13}^{2} = f_{23}^{1} = f \)
- Single elliptic T-dual, non-geometric space:

\[ H_{123} = Q_{3}^{12} = H \]
- Double elliptic, genuinely non-geometric space:

\[ H_{123} = Q_{3}^{12} = H , \quad f_{13}^{2} = f_{23}^{1} = f \]
E.g. double elliptic background:

\[
\tau(x_3) = \frac{\tau_0 \cos(f x_3) + \sin(f x_3)}{\cos(f x_3) - \tau_0 \sin(f x_3)}, \quad f \in \frac{1}{4} + \mathbb{Z},
\]

\[
\rho(x_3) = \frac{\rho_0 \cos(H x_3) + \sin(H x_3)}{\cos(H x_3) - \rho_0 \sin(H x_3)}, \quad H \in \frac{1}{4} + \mathbb{Z}.
\]

\[
\Rightarrow \quad \tau(2\pi) = -\frac{1}{\tau(0)}, \quad \rho(2\pi) = -\frac{1}{\rho(0)}
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\]

\[\implies \tau(2\pi) = -\frac{1}{\tau(0)} , \quad \rho(2\pi) = -\frac{1}{\rho(0)} \]

Patching is generated by the coordinate transformation:

\[
\chi(x^1, x^2, \tilde{x}_1, \tilde{x}_2) = \frac{1}{2} (x^1 x^2 + \tilde{x}_1 \tilde{x}_2 - \tilde{x}_1 x^2 - x^1 \tilde{x}_2) - \frac{3}{2} (\tilde{x}_1 x^1 + \tilde{x}_2 x^2) - \frac{1}{4} ((x^1)^2 + (x^2)^2 + (\tilde{x}_1)^2 + (\tilde{x}_2)^2) .
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\]

\[
- \frac{1}{4} ((x^1)^2 + (x^2)^2 + (\tilde{x}_1)^2 + (\tilde{x}_2)^2).
\]

Corresponding Killing vectors of background:

\[
K^j_i = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{2}(Hx^3 + f\tilde{x}^3) & \frac{1}{2}(Hx^2 + f\tilde{x}^2) & 0 \\
0 & 0 & 1 & 0 & 0 \\
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E.g. double elliptic background:

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\[ -\frac{1}{4} \left( (x^1)^2 + (x^2)^2 + (\tilde{x}_1)^2 + (\tilde{x}_2)^2 \right). \]

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0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Killing vectors do not satisfy strong constraint. However their algebra closes!
V) Outlook & open questions
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- Non-commutative & non-associative closed string geometry arises in the presence of non-geometric fluxes (like open string non-commutativity on D-branes with gauge flux). This leads to a non-associative tri-product (like the star-product).
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