Some Comments on Kerr/CFT
and beyond

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Outline

- **The Big Picture**: extremal limit of general black holes.

- **Microscopics of Kerr/CFT**: proposal for precision counting.

- **AdS$_2$ Quantum Gravity**: Kerr/CFT as the theory of (some) diffeomorphisms.

- Some references:
  K. Hanaki and FL (in progress).
Extremal Black Holes: Overview

- **Setting:**
  Black holes in $D = 4$ SUGRA, single center, asymptotically flat, $N = 8, 4, 2$ theory. Parameters $(M, J, Q_I, P^I)$.

- **Extremal limit:**
  $T_H = 0$.
  There is an $\text{AdS}_2$ factor in geometry.

- Distinguish **3** types of extremal Black Holes:
  i) BPS.
  ii) non-BPS extremal.
  iii) extremal Kerr.
Example 1

- **Theory:** M theory on $CY \times S^1$

- **Black hole:**
  \[ M5 \text{ on } P \times S^1 \text{ (} P \text{ a divisor)} \text{ with } P^3 \neq 0 \text{ ("3 charges")} \]
  Momentum = $n$ along $S^1$ (so 4 charges total)

- Black hole entropy: $S = 2\pi \sqrt{|n| P^3}$

- **Two extremal limits:**
  BPS: $n > 0$. The "fourth" charge break same SUSYs as $P$.
  non-BPS: $n < 0$. The "fourth" charge break opposite SUSYs so none are left.
Example 2: Kerr-Newman

- **Theory:** Einstein-Maxwell, ”diagonal charges”, $Q^4 \sim n P^3$

- **Black hole: Kerr-Newman**
  
  Black hole entropy ($G_4 = 1$):
  
  $$S = 2\pi \left[ (M^2 - \frac{1}{2}Q^2) + \sqrt{M^2(M^2 - Q^2) - J^2} \right] = S_L + S_R$$

- **BPS:** $M^2 = Q^2$, $J = 0$, $S = 2\pi \cdot \frac{1}{2} Q^2$

- **Extremal Kerr:** $Q^2 = 0$, $M^4 = J^2$, $S = 2\pi |J|$

- **BPS and Kerr both correspond to R in a specific state, L carries entropy**
The Big Picture

The *general* black hole — with parameters: \((M, J, Q_I, P^I)\) — is described by a 2D CFT with L and R chiralities that interact weakly:

\[
S = S_L + S_R, \quad \beta_H = \frac{1}{2}(\beta_L + \beta_R)
\]

R-movers have the ability to carry \(J\).

**BPS**: \(T_R \rightarrow 0\) (with \(J = 0\)). R-movers in ground state, J-carryers not excited. L-movers carry entropy.

**Extremal Kerr**: \(T_R \rightarrow 0\) (with \(J \neq 0\)). R-movers in a definite state, with J-carryers excited. L-movers carry entropy.

**General Black Hole**: direct product of BPS (L) and non-BPS theory (R), with level matching relating the two chiralities.
Working assumption: the entropy of extremal Kerr comes from the "same" states (L-movers in our convention) as the BPS entropy.

The difference: the R-movers, for Kerr in a state that breaks SUSY spontaneously, instead of the SUSY preserving ground state.

Strategy for precision counting: consider the CFT underlying the BPS counting. Keep the dynamical chirality (holomorphic=L-movers) intact, but modify the inert chirality (anti-holomorphic=R-movers) by spectral flow.
Setting

The $D1/D5$ on $K3 \times S^1$, described by the sigma-model on

$$\mathcal{M}^k / \Sigma_k$$

with $\mathcal{M} = K3$, $k = n_1 n_5 + 1$. The central charge is $c = 6k$.

Excitations at level $h = p$ give asymptotic degeneracy

$$S = 2\pi \sqrt{\frac{ch}{6}} = 2\pi \sqrt{kp}$$

The 4D version of model involves adding a KK-monopole: basic reasoning remains, but some details change.
5D Counting

Vertex operators:

\[ \mathcal{V}(z, \bar{z}) = \mathcal{V}^L_{\text{int}}(z)e^{iF_L \varphi_L(z)/\sqrt{2k}} \cdot \mathcal{V}^R_{\text{int}}(\bar{z})e^{iF_R \varphi_R(\bar{z})/\sqrt{2k}} \]

Bosonized the R-currents: \( J = \sqrt{2k} \partial \varphi_L, \bar{J} = \sqrt{2k} \bar{\partial} \varphi_R \)

Spacetime angular momentum: \( F_{R,L} = 2j_{R,L} \)

Conformal weights:

\[ h_R = h^\text{int}_R + \frac{1}{4k} F_R^2 \]
\[ h_L = h^\text{int}_L + \frac{1}{4k} F_L^2 \]

Momentum of the state:

\[ p = h_L - h_R \]
**Extremal limit:**

\[ h^{\text{int}}_R = 0 , \quad \Rightarrow \quad h_R = \frac{1}{4k} F^2_R \quad \text{(extremal)} \]

**Origin of entropy:** freedom in \( \mathcal{V}^L_{\text{int}}(z) \) with weight

\[ h^{\text{int}}_L = h_L - \frac{1}{4k} F^2_L = p + h_R - \frac{1}{4k} F^2_L = p + \frac{1}{4k} F^2_R - \frac{1}{4k} F^2_L \]

The **black hole entropy**:

\[ S = 2\pi \sqrt{\frac{c h}{6}} = 2\pi \sqrt{k p + j^2_R - j^2_L} \]

BMPV black hole: special case \( j_R = 0 \).

Extremal 5D Kerr: \( j_L = p = 0 \) with entropy

\[ S = 2\pi |j_R| \]
• The *4D version of computation*: add KK-monopole $\Rightarrow$ SUSY broken in L-sector $\Rightarrow$ there is no $j_L$. But $j_R$ identified with $4D$ angular momentum.

• **Uncharged case**: $n_1 = n_5 = 0 \Rightarrow k = 1$ a singular limit so computation not justified.

• **Answer analysis** (inconclusive): for $p = 0$ the level $k$ cancels so entropy would work out for Kerr no matter what central charge is claimed.
Precision Counting

The **partition function** in the RR sector (with \((-)^F\) inserted):

\[
Z(\tau, z, \bar{\tau}, \bar{z}) = \text{Tr}[(-)^F y^F_L q^{L_0 - \frac{c_L}{24}} y^F_R \bar{q}^{\bar{L}_0 - \frac{c_R}{24}}]
\]

The **elliptic genus**: take \(\bar{z} = 0\) \(\Rightarrow\) \(Z(\tau, \bar{\tau}, z, 0)\) is independent of \(\bar{\tau}\).

Combine with spectral flow to map out the dependence on **one** anti-holomorphic parameter:

\[
Z(\tau, z, \bar{\tau}, \ell \bar{\tau} + \bar{m}) = \bar{q}^{-\frac{1}{k}j_R^2} \times \text{holomorphic}
\]
Master Partition Function

Precision counting is best summarized in terms of an ensemble with arbitrary number \((m)\) strings: a sum over CFTs, with level \(k = m \Rightarrow\) full (4D) partition function in sector with angular momentum \(j_R\):

\[
Z_{j_R}(\tau, z, \sigma) = -64 \frac{1}{qy} \prod_{b=0}^{1} \prod_{l,m,j} (1 - q^l q^{\frac{1}{m} j_R^2} p^m y^j) c_b(4lm - j^2)
\]

Asymptotic behavior of the entropy

\[
S \sim 2\pi \sqrt{j_R^2 + \vec{q}^2 \vec{p}^2 - (\vec{p}\vec{q})^2}
\]

for any charges \((\vec{q}, \vec{p})\) where the argument of the square root is positive.

Clearly: partition function encodes many subleading corrections which we are working out.
The modular behavior is confusing (and perhaps not right).

*This work is still in progress*
Extreme Kerr Quantum Geometry

Change focus: determine features of the quantum theory from study of the spacetime geometry.

(Near) extreme Kerr: warped AdS$_2$ geometry.

Goal: analyze 2D diffeomorphisms with AdS$_2$ boundary conditions.

Result: compute $c_R = 12J$ from strictly 2D point of view.

Establish consistency with 3D Brown-Henneaux-Strominger treatment of the BTZ black hole $\Rightarrow$ gain confidence in results.

Interpretation in Kerr setting: AdS$_2$ quantum gravity is theory is excitations above extremality.
Near extremal limit directly in Kerr geometry (including excitation energy):

\[ ds^2 = \frac{1 + \cos^2 \theta}{2} \left[ -\frac{U^2 - \epsilon^2}{\ell^2} dt^2 + \frac{\ell^2}{U^2 - \epsilon^2} du^2 + d\theta^2 \right] + \ell^2 \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} (d\phi + \frac{U}{\ell^2} dt)^2 \]

This is a *warped AdS*$_2$.

**2D Black Hole:**

\[ ds_2^2 = -\frac{U^2 - \epsilon^2}{\ell^2} dt^2 + \frac{\ell^2}{U^2 - \epsilon^2} du^2 + d\theta^2 . \quad \mathcal{B}_t = \frac{U}{\ell^2} dt \]

In *classical theory*: diffeomorphic to AdS$_2$.

In *quantum theory*: an excited sector of AdS$_2$ quantum gravity.
2D Effective Theory

Effective 2D theory = gravity+$U(1)$ gauge theory

\[ S_{\text{Kerr}} = \frac{\ell^2}{4G_4} \left[ \mathcal{R}^{(2)} + \frac{1}{\ell^2} - \frac{\ell^2}{2} G_{\mu\nu} G^{\mu\nu} \right] \]

Notation: appropriate for reduction from 4D Kerr ($G_4 = 4\pi G_2 \ell^2$ is more neutral). The scalar ($\psi$) is taken to vanish (by asymptotic e.o.m.)

Boundary terms guarantee well defined variational principle

\[ S_{\text{bndy}} = \frac{\ell^2}{2G_4} \int dt \sqrt{-\gamma} \left[ \mathcal{K} - \frac{1}{2\ell} + \frac{\ell}{2} B_a B^a \right] \]
Boundary Conditions

General 2D spacetime (in convenient gauge)

\[ ds^2 = d\rho^2 + h_{tt} dt^2, \quad B = B_t dt \]

Asymptotically AdS_2 boundary conditions

\[ h_{tt} = h^{(0)} e^{2\rho/\ell} + h^{(2)} + h^{(4)} e^{-2\rho/\ell} + \ldots \]
\[ B_t = B^{(0)} e^{\rho/\ell} + B^{(2)} + B^{(4)} + \ldots \]

satisfying the constraint: \[ h^{(0)} + \ell^2 B^{(0)^2} = 0. \]

Diffeomorphism+gauge transformations preserving the gauge conditions

\[ \epsilon^t = \xi(t) - \frac{\ell^2}{2h^{(0)}} \partial_t^2 \xi e^{-2\rho/\ell} + O(e^{-4\rho/\ell}) \]
\[ \epsilon^\rho = -\ell \partial_t \xi(t) \]
\[ \Lambda = -\frac{\ell}{h^{(0)}} \partial_t^2 \xi (B^{(0)} e^{-\rho/\ell} + \frac{1}{2} B^{(2)} e^{-2\rho/\ell}) + O(e^{-3\rho/\ell}) \]
2D Effective Theory

Key computation in standard (higher dim) AdS/CFT: the energy momentum tensor, and other linear response functions.

The *linear response functions* of the 1D boundary theory:

\[
T_{ab} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = -\frac{\ell^2}{4G_4}(\frac{1}{\ell} \gamma_{tt} + \ell B_t^2)
\]

\[
J_t = \frac{1}{\sqrt{-\gamma}} \gamma_{tt} \frac{\delta S}{\delta B_t} = \frac{\ell^3}{2G_4}(-n^\nu g_{\mu t} + B_t)
\]
Generator of Diffeomorphisms

Variation under a diffeomorphism:

\[ \delta_\epsilon S = \frac{1}{2} \int dt \sqrt{-\gamma} T^{ab} \delta_\epsilon \gamma_{ab} + \int dt \sqrt{-\gamma} J^a \delta_\epsilon B_a \]

so generator of diffeomorphisms is not just em-tensor:

\[ Q_\epsilon = \sqrt{-\gamma} \gamma^{tt}(T_{tt} + J_t B_t) \]

Second term not suppressed in 2D (gauge field linearly rising).

Expression gives correct notion of local energy

\[ T_{tt} + B_t J_t = \frac{\ell}{4G} \frac{\epsilon^2}{\ell^2} \]

Boundary energy (including measure factors) vanishes in strict extremal limit, as we expect for R-excitations; but scaling limit gives finite notion of excitation energy.
Transformation to AdS$_2$

The AdS$_2$ black hole is related to AdS$_2$ in Poincare coordinates

$$ds^2 = \frac{\ell^2}{y^2}(dy^2 - dw^2), \quad B = -\frac{\ell}{y}$$

through the coordinate transformation

$$y = \frac{\ell \epsilon}{\sqrt{U^2 - \epsilon}} e^{-\epsilon t/\ell^2}$$

$$w = \frac{\ell U}{\sqrt{U^2 - \epsilon}} e^{-\epsilon t/\ell^2}$$

$$\Lambda = -\frac{1}{2} \ln \left( \frac{U + \epsilon}{U - \epsilon} \right)$$

Comment: coordinate transformation singular at $U = \epsilon$
More important comment: near $U \to \infty$ the coordinate change is $w = \ell e^{-\epsilon t/\ell^2}$, corresponding to Rindler space with $T = \frac{\epsilon}{2\pi \ell^2}$.

The generator of diffeomorphisms realizes the Virasoso algebra

$$T_{tt} + B_t J_t = \frac{c}{12} \ell \{ w, t \} (\partial_w t)^{-2} = \frac{\pi^2}{6} c \ell T^2$$

Comparing with the explicit computation for the AdS$_2$ black hole, we infer the central charge

$$c = \frac{6 \ell^2}{G_4}$$

For Kerr $G_4 = 2 \ell^2 J \Rightarrow c_{\text{Kerr}} = 12 J$.

Note: derivation only makes reference to 2D.
The 3D lift: local part

AdS$_2$ results are consistent with standard results in 3D.

Embedding of AdS$_2$ into AdS$_3$:

\[ ds^2_3 = \ell^2 (d\theta + \mathcal{B}_t dt)^2 + d\rho^2 + h_{tt} dt^2. \]

The 2D diff+gauge transformations that preserve AdS$_2$ boundary conditions is a genuine subgroup of diff’s that preserve AdS$_3$ boundary conditions.

The 2D Virasoro is identified with the $SL(2, R)_R$ of the 3D theory.

The central charges match precisely, and the normalizations of energy match.
The 3D lift: global part

The 3D lightcone coordinates $w^\pm = \phi \pm t$ are identified as

$$w^- = \frac{r_+ + r_-}{8\ell^2} t$$

$$w^+ = \frac{2\ell}{r_+ + r_-} \theta - \frac{r_+ - r_-}{8\ell^2} t$$

Note: boundary conditions are not mapped correctly (e.g. $w^-$ is periodic, $t$ is not).

Remedy: the DLCQ limit

$$w^- \rightarrow \lambda^{-1} w^- , \quad e^{2\eta} \rightarrow \lambda e^{2\eta} , \quad r_+ - r_- = 4\lambda \epsilon$$

After DLCQ, the boundary conditions work out and the solution is tuned so the (infinitesimal) excitation energy is finite

$$w^- = \frac{t}{2\ell} , \quad w^+ = \frac{\theta}{2} - \frac{\epsilon}{2\ell^2 t}$$
Isometries

The $2\pi$ identification on the azimuthal coordinate $\phi$ correspond to identifications on the lightcone coordinates $w^\pm$.

The identification on $w^+$ is a hyperbolic holonomy that breaks $SL(2)_L \rightarrow U(1)_L$, as it does for BTZ.

After the DLCQ limit there is no identifications on $w^- \sim t$ so $SL(2)_R$ is preserved. This group is the isometry group AdS$_2$.

The broken $U(1)_L \in SL(2)_L$ is enhanced to a Virasoro algebra, that acts on the excitations.
Phenomenology of Kerr Entropy

Take near extremal limit of general Kerr entropy:

\[ S = 2\pi \left( M^2 + \sqrt{M^4 - J^2} \right) \to 2\pi \left( \frac{c_L}{12} + \sqrt{\frac{c_R h_R}{6}} \right) \]

Central charges: \( c_L = c_R = 12J \)

Level (energy) of right movers:

\[ h_R = \frac{\epsilon^2 \lambda^2}{2l_P^2} , \quad \epsilon \lambda \equiv \frac{1}{2}(r_+ - r_-) \]

Original Kerr/CFT: extremal Kerr = ground state of chiral (L only) CFT

Here: focus on the R-sector, \( c_R \) and excitation energy.
Kerr/CFT from AdS$_2$

The central charge is reproduced from AdS$_2$ quantum gravity.

The “phenomenological” energy has the right form to match a CFT we need to account for its value

$$E = \frac{\pi^2}{6} c T_H^2 R$$

The conformal weight is finite after the DLCQ limit

$$h = \frac{ER}{\lambda} = \frac{c}{24} \cdot \frac{2\epsilon^2}{\ell^2}$$

First attempt on the coefficient: infer the value $R = \sqrt{2}\ell$ from our computations. But this value is arbitrary in the CFT (C is for conformal).
Better: instead we can match the scale invariant ratio:

\[
\frac{E}{R\lambda} = \frac{\pi^2}{6} cT^2 = \frac{T_{tt} + B_t J_t}{\ell}
\]

This works out quantitatively!

After the DLCQ rescaling \( h \gg c \) in the black hole regime so Cardy’s formula applies.
Summary

Aspects of Kerr/CFT:

- Relatively firm understanding of origin as a subgroup of diffeomorphism invariance.
- Beginnings of precision counting.
- Hints of a structure that generalizes to all black holes.