Spherical excision for moving black holes and summation by parts for axisymmetric systems

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Outline

- Introduction
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  - Overlapping grids

Axisymmetric scalar field on a black hole background
  - Single grid energy preserving discretization
  - Spherical excision of a boosted black hole

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Motivation: Cubical Excision

Excision: an outflow inner boundary eliminates the singularity.

For a Schwarzschild BH in KS coordinates the side length of the cube must be smaller than $4\sqrt{3}M/9$. 
Motivation: Cubical Excision

For a rotating BH in KS coordinates cubical excision is not possible if $a \gtrsim 0.0851M$. 
Hyperbolic initial-boundary value problem in a domain $\Omega$ with a moving hole.
Single vs. Overlapping Grids

**Single Cartesian grid**
- **Pros**
  - Stability proofs
  - Easier to parallelize
- **Cons**
  - Cubical domains,
  - complicated algorithms,
  - uniform grid

**Overlapping grids**
- **Pros**
  - Smooth, time dep. boundaries
  - Well-posedness proofs for hyperbolic problems in general domains
- **Cons**
  - Interpolation
  - Few proofs of stability
We write $\sqrt{-g} \nabla_\mu \nabla^\mu \Phi = \partial_\mu (\gamma^{\mu\nu} \partial_\nu \Phi) = 0$ in first order form

$$\begin{align*}
\partial_t \Phi &= T, \\
\partial_t T &= - (\gamma^{ti} \partial_i T + \partial_i (\gamma^{it} T) + \partial_i (\gamma^{ij} d_j) + \partial_t \gamma^{tt} T + \partial_t \gamma^{ij} d_j) / \gamma^{tt}, \\
\partial_t d_i &= \partial_i T.
\end{align*}$$

The constraint variables $C_i \equiv d_i - \partial_i \Phi$ propagate trivially.

Axisymmetry: $\partial_\phi$ is a spacelike Killing field $\rightarrow \partial_\phi \gamma^{\mu\nu} = 0$.

Eliminate $\Phi, d_\phi$: $3 \times 3$ system. Introduce $u^T = (T, d_1, d_2)$ and write

$$\partial_t u = A^i(t, \vec{x}) \partial_i u + B(t, \vec{x}) u.$$
The Energy Method

In coordinates adapted to a timelike Killing field $\partial_t$, the energy

$$E = \int_{\Omega} u^T H u \, d^2 x = \int_{\Omega} (-\gamma^{tt} T^2 + \gamma^{ij} d_i d_j) \, d^2 x$$

is conserved: $\partial_t E = \int_{\partial \Omega} (w_{\text{in}}^2 - w_{\text{out}}^2) \, d \sigma$.

In KS coordinates the integrand is positive definite where $\partial_t$ is timelike.

The energy method. Symmetrizable hyperbolic systems with maximal dissipative boundary conditions

$$w_{\text{in}}(t, \vec{x}) = S w_{\text{out}}(t, \vec{x}) + g(t, \vec{x}), \quad \vec{x} \in \partial \Omega$$

Gives sufficient conditions for well posedness.
Energy conserving discretization \((E = h_1 h_2 \sum_{ij} u^T_{ij} H_{ij} u_{ij} \sigma_{ij})\):

\[
\begin{align*}
\partial_t T &= - (\gamma^{ti} D_i T + D_i (\gamma^{it} T) + D_i (\gamma^{ij} d_j) + \partial_t \gamma^{tt} T + \partial_t \gamma^{ij} d_j) / \gamma^{tt}, \\
\partial_t d_i &= D_i T.
\end{align*}
\]

where \((u, Dv)_h = - (Du, v)_h + u_j v_j |_0^N\).

Boundary data at corners
Consider 2D wave equation in polar coordinates

\[ \partial_t T = \frac{1}{\rho} \partial_\rho (\rho P) ; \quad \partial_t P = \partial_\rho T \]

Use regularity conditions on the axis. The semidiscrete system

\[
\begin{align*}
\partial_t T_i &= \begin{cases} 
2D_+ P_0 & i = 0 \\
\frac{1}{\rho_i} D_0 (\rho P)_i & i \geq 1
\end{cases} \\
\partial_t P_i &= D_0 T_i, \quad i \geq 1
\end{align*}
\]

conserves the discrete energy

\[
E = \sum_{i=1}^{+\infty} (T_i^2 + P_i^2) \rho_i \Delta \rho + \frac{1}{4} T_0^2 \Delta \rho^2
\]
Spherical Excision

Introduce a spherical grid adapted to the event horizon
Boosted Black Hole

- Boosted cylindrical KS coordinates \( \{ \bar{t}, \bar{\rho}, \bar{z} \} \) on the (base) cylindrical grid

\[
\bar{t} = \gamma(t - \beta z) \quad \bar{z} = \gamma(z - \beta t)
\]

- Co-moving spherical coordinates \( \{ t', r', \theta' \} \) on the spherical grid

\[
t' = \gamma(t - \beta z)
\]

- Communication done via interpolation of all fields, followed by the transformation law for 1-forms:

\[
\frac{\partial \Phi}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial \Phi}{\partial x^\nu}
\]
Boosted Black Hole ($\beta = -0.75$)
Numerical Experiments

- **Boosted BH background:**

  - $\beta = -0.75$;
  - $Q = \log_2 \frac{\|v_h - u_{\text{exa}}\|}{\|v_{\text{h}/2} - u_{\text{exa}}\|}$;
  - Cylindrical $256 \times 512$
    - $(\rho, z) \in [0, 10] \times [-10, 10]$;
  - Spherical $128 \times 384$
    - $(r, \theta) \in [2, 4] \times [0, \pi]$;
  - 4RK; $\frac{k}{h} = 3/4$;
  - dissip: 0.02.

- Go to animations
Numerical Experiments

- Minkowski background:
  - No boost ($\beta = 0$);
  - $\varepsilon_h = \|v_h - u_{\text{exa}}\|_h$;
  - Cylindrical $90 \times 170$
    $$ (\rho, z) \in [0, 10] \times [-10, 10]; $$
  - Spherical $50 \times 68$
    $$ (r, \theta) \in [1, 5] \times [0, \pi]; $$
  - dissip: 0.02.
- The interpolation between the grids does not introduce growth in the error.
Conclusions

- Cubical excision has severe limitations.
- Excising the excisable can only help.
- Overlapping grids can be adapted to the geometry of the problem; allow for moving boundaries; closer to the continuum problem.
- Experiments with axisymmetric scalar field in boosted black hole background are very encouraging.
- Currently working on rotating case, higher order accuracy, dynamical tracking of excisable region, etc.
- Our belief is that if one cannot evolve a scalar field, then one cannot evolve a black hole.