EH Tracking: Foundations and Applications

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1. Motivation:
   (a) Black holes in general
   (b) Applications to problems in Numerical Relativity

2. Foundations: The Eikonal equation.

3. Numerical Methods
   (a) Comoving Front Tracking.
   (b) Level Set Viscosity Solutions

Further Information

- Caveny and Matzner gr-qc/0303109
- Anderson, Caveny and Matzner gr-qc/0303099
GENERATED by NULL GEODESICS with NO FUTURE END POINT.

1. Generators lie in EH for at least some finite lapse of affine parameter
2. Followed into the future, generators cannot leave EH
3. Followed into the past, generators leave EH at caustic
4. Unique generator through each non-caustic event
Motivations for This Research

EH TRACKING BELONGS IN NR TOOL BOX

- Phenomenology: Topology changes of EH (Teukolsky / Winicour / etc), differentiability (Wald / Chrusciel), creases, caustics, etc.

- BBHC: Qualitative and quantitative understanding masses, areas, angular momentum, merger time, future null infinity, etc.

\[ \delta M = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J \] (1)
Our Primary Results

- Numerical Methods:
  - Comoving Front Tracking Method (2+1)
  - Viscosity Solution Level Set Method (3+1)

- Phenomenology:
  - Fluid droplet analogy & power law scaling of minimal throat connecting objects at merger / birfurcation
  - First numerical analysis of EH in asymmetric BBHC.
Canonical generator for null geodesics:

\[
\partial_\tau s = -H (x^a, \partial_a s) = -\partial_a s g^{ab} \partial_b s, \tag{2}
\]

where

\[
H (x^a, p_a) = p_a g^{ab} p_b = \dot{x}^a g_{ab} \dot{x}^b = L (x^a, \dot{x}^a) \tag{3}
\]

\[
s = \int d\tau L (x^a, \dot{x}^a). \tag{4}
\]
Seperability

\[ s = \omega \tau - S \left( x^a \right) \quad (5) \]

so for \( \omega = 0 \)

\[ \partial_a S g^{ab} \partial_b S = 0. \quad (6) \]

IBVP Solution:

\[ \partial_t S = \frac{1}{g_{tt}} \left( -g^{ti} \partial_i S \pm \sqrt{(g^{ti} \partial_i S)^2 - g^{tt} g^{ij} \partial_i S \partial_j S} \right) \quad (7) \]
\( g_{4t}^{tt} = -\alpha^{-2}, \ g_{4t}^{ti} = \alpha^{-2} \beta^i, \ g_{4i}^{ij} = \gamma^{ij} - \alpha^{-2} \beta^i \beta^j \)  \hspace{1cm} (8) \\

\[ \partial_t S = \beta^i \partial_i S \pm \alpha \sqrt{\partial_i S \gamma^{ij} \partial_j S} \]  \hspace{1cm} (9) \\

\[ H(t, x^i, p_i) = -\beta^i p_i \pm \alpha \|p\|. \]  \hspace{1cm} (10) \\

\[ p_i = \partial_i S. \]  \hspace{1cm} (11)
The Eikonal and Null Geodesics

Corresponding particle problem:

\[ \partial_t S = \beta^i \partial_i S \pm \alpha \sqrt{\partial_i S \gamma^i j \partial_j S} \]  \hspace{1cm} (12)

\[ \dot{x}^i = -\beta^i \pm \alpha \frac{p^i}{\|p\|}, \quad \dot{p}_i = -\partial_i H \]  \hspace{1cm} (13)

\[ p_i = \partial_i S. \]  \hspace{1cm} (14)
The Eikonal ⇔ Null Geodesics

- Identifies $\mathcal{S}$ as a Riemann invariant of NG’s.
- Provides convenient translation to NG methods.
- Identifies blow up of $\mathcal{S}$ where NG’s cross.
Black Hole Signatures

- $t \to \infty$, OND (interior or exterior) propagate away from horizon to future asymptotic states. \( \Rightarrow \) Detection Unstable $t_\uparrow$.
- $t \to -\infty$, OND (interior or exterior) approach horizon to arbitrary precision. \( \Rightarrow \) Detection Stable $t_\downarrow$?
- Since $S$ is a Riemann invariant, OND propagated into the past become MULTI-VALUED on EH; e.g., $S$ is discontinuous, or gradient becomes arbitrarily large.
Let $C^i(t)$ be average value of points on $S(\Gamma) = 0$; e.g., track a single level set or *front*

$$S = r - U(t, \theta, \phi) = 0 \quad (15)$$

$$\partial_t U = \beta^r + \beta^I \partial_I U + \rho \quad (16)$$

$$\rho = \alpha \sqrt{\gamma^{rr} + 2 \gamma^{rI} \partial_I U + \partial_I U \gamma^{IJ} \partial_J U}. \quad (17)$$

$C^i(t) \rightarrow C^i(t + dt) \iff$ comoving grid for $U$; e.g.,

$G^n \rightarrow G^{n+1}$. 
Pseudo Code

- Load $x^{i,n}(M,N)$
- $x^i(N,M) \rightarrow C^i, U(M,N)$
- Interpolate onto $\Gamma$
- Update $U^n \rightarrow U^{n+1}$
- Update $C^n \rightarrow C^{n+1}$
- Pass data $x^{i,n+1}(M,N) \rightarrow x^{i,n+1}(M,N)$
- Return $x^{i,n+1}(M,N)$
\( g_{ab} = \eta_{ab} + 2Hl_\alpha l_\beta \) \hspace{1cm} (18)

\[
l_\alpha = \left( 1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r} \right), \quad H = \frac{Mr^3}{r^4 + a^2z^2} \hspace{1cm} (19)
\]

\[
\frac{x^2}{r^2 + a^2} + \frac{y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \hspace{1cm} (20)
\]

- Test Accuracy of detection / Areas.
- Obtain ‘rule of thumb’ for relaxation time.
• Test accuracy of data passing

1. Shifted source \( x \rightarrow x + b \)

2. ‘Wobbling’ source

\[
x \rightarrow x' = x + b \cos \omega t, \tag{21}
\]

\[
y \rightarrow y' = y + b \sin \omega t, \tag{22}
\]
Application: Kastor Traschen

- Dynamic application: Merging black holes in deSitter background.

\[ ds^2 = -U^2 dt^2 + \frac{1}{U^2} (dx^2 + dy^2 + dz^2) \]  \hspace{1cm} (23)

\[ U = -t\sqrt{\Lambda/3} + \frac{M_1}{r_1} + \frac{M_2}{r_2} \]  \hspace{1cm} (24)

- Identifies breakdown of method at merger
- Identifies scaling of minimal throat at bifurcation.
• Provides analysis of holes stationary in asymptotic past and future.
• No special treatment of discontinuity required.
• No special treatment of masks.
• Away from breakdown, highly accurate method.
Level Set Viscosity Solutions

- Integrate $\partial_t S = -H \left( t, x^i, \partial_j S \right)$ directly
- Track one level set $S(\Gamma) = 0$ via update of $S$.
- Regulate formation of discontinuity with explicit viscosity

\[
\partial_t S = -H \left( t, x^i, \partial_j S \right) \quad (25)
\]

\[
\partial_t \phi = \epsilon \nabla^2 \phi - H \left( t, x^i, \partial_j \phi \right) \quad (26)
\]
Error Analysis

- Variety of methods available for regulating solution singularity; e.g., Kreiss Oliger dissipation, High Res. CFD methods
- Explicit second order term is simple, and provides understanding / control of viscosity: e.g., WKB

\[ \phi \approx A \exp(-S/\epsilon) ; \]

\[ \phi (t, x^i) = S (t, x^j) + \mathcal{O} (\epsilon) \quad (27) \]

Improved viscosity solution:

\[ \phi_I = \phi_\epsilon + (\phi_\epsilon - \phi_2\epsilon) + \mathcal{O} (\epsilon^2) \quad (28) \]
Advantages

- No interpolation.
- Requires only a metric as input.
- Reduces scope of surveys via symmetries.
- No special coordinates.
- Handles arbitrary numbers of sources.
- Continuously monitors arbitrary topological transitions.
- $2 \frac{a}{M} = 1/2, M = 1$ black holes.
- Centers at $(\pm 6, \pm 2, 0)$.
- Boosted at $v_x = \pm 1/2c, v_y = v_z = 0$.
- Grid with $N^3$ points $N = 121$
- Run length $10M$
Initial Data for Eikonal

- AH trackers find merger time about $t = 8M$
- Quasi normal ringing requires $20M$
- EH tracker requires at least $t = 4M$ per $e$-folding.
- EH signature suggests iterating $\Gamma$ data over final time slice for $\mathcal{O}(10) - \mathcal{O}(100)$ $e$-foldings to extract initial data for Eikonal.
- Augment resulting data $\Gamma_i$ with analysis of AH’s, and with area analysis to extract final EH.
Error Analysis

- Analysis of exact solutions shows error of area extractions < 10%
- Analysis of viscosity surveys shows error of areas ≈ 7% and merger time increasing with increasing viscosity.
\[ a/M = 0.5 \]

\[ a/M = 0.75 \]

\[ a/M = 0.25 \]

\[ a/M = 0.0 \]
Search for the Signature

- Signature cannot be seen in such a short data set, primary source of error.
- Find lower bound of $A \approx 74.5 \pm 6.0M$
- Corresponds to masses of about $M = 1.36$ at
- Merger time of about $t^* = 1.8M$
Conclusions

- Lay foundations for tracking black holes in numerical relativity
- Two methods: Comoving Mesh method, Level Set Viscosity Method.
- Identified an analogy to fluid droplet bifurcation
- First analysis of asymmetric binary black hole coalescence including: Merger time analysis, Area analysis, and study of the nature of the topological transition.