Tensor modes in loop quantum cosmology
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Aims of the talk

To compute cosmological observables from LQC linear tensor perturbations.

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where

\[ \rho_c \equiv \frac{3}{8\pi G\gamma^2 \bar{\mu}^2 p} \propto a^{-2(1-2n)}. \]
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\[ \alpha = \frac{1 + n}{3r} \lambda \left( \left| 1 + \frac{1}{\lambda} \right|^{\frac{3r}{2(1+n)}} - \left| 1 - \frac{1}{\lambda} \right|^{\frac{3r}{2(1+n)}} \right), \quad \lambda \sim \mathcal{V}^{2(1+n)/3} \]
Two regimes (well-defined in inhomogeneous patches)
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Quasi-classical regime: large volumes ($\lambda \gg 1$)

\[
\alpha_c \approx 1 + \left[3r^2 \left(1 + n\right) - 1\right]^{\frac{1}{6}} \lambda^2 \equiv 1 + \alpha_c (\sqrt{\Delta a})
\]

where $\alpha_c = 4 \left(1 + n\right)$, $\alpha_c \approx -1.162 < \alpha_c < 1.9 \approx 0.1$.
Two regimes (well-defined in inhomogeneous patches)

Quasi-classical regime: large volumes ($\lambda \gg 1$)

$$\alpha \approx 1 + \left[ \frac{3r}{2(1+n)} - 2 \right] \left[ \frac{3r}{2(1+n)} - 1 \right] \frac{1}{6\lambda^2}$$
Two regimes (well-defined in inhomogeneous patches)

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\alpha \approx 1 + \left[ \frac{3r}{2(1+n)} - 2 \right] \left[ \frac{3r}{2(1+n)} - 1 \right] \frac{1}{6\lambda^2} \\
\equiv 1 + \alpha_c \left( \frac{\sqrt{\Delta}}{a} \right)^c
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Two regimes (well-defined in inhomogeneous patches)

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$$\equiv 1 + \alpha_c \left( \frac{\sqrt{\Delta}}{a} \right)^c$$

where

$$c = 4(1 + n), \quad \alpha_c = \frac{[3r - 4(1 + n)][3r - 2(1 + n)]}{3^42} \left( \frac{\Delta_{Pl}}{\Delta} \right)^2.$$
Two regimes (well-defined in inhomogeneous patches)

Quasi-classical regime: large volumes ($\lambda \gg 1$)

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\alpha \approx 1 + \left[ \frac{3r}{2(1+n)} - 2 \right] \left[ \frac{3r}{2(1+n)} - 1 \right] \frac{1}{6\lambda^2} 
\]

\[
\equiv 1 + \alpha_c \left( \frac{\sqrt{\Delta}}{a} \right)^c
\]

where

\[
c = 4(1+n), \quad \alpha_c = \frac{[3r - 4(1+n)][3r - 2(1+n)]}{3^4 2} \left( \frac{\Delta_{\text{Pl}}}{\Delta} \right)^2.
\]

Assuming $\Delta = \Delta_{\text{Pl}}$

\[
4 < c \leq 6, \quad -0.01 \approx -\frac{1}{162} < \alpha_c < \frac{1}{9} \approx 0.1
\]
Two regimes (well-defined in inhomogeneous patches)
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Near-Planckian regime ($\lambda \ll 1$)
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Near-Planckian regime $(\lambda \ll 1)$

$$\alpha \approx \lambda^2 \frac{3r}{2(1+n)}$$
Two regimes (well-defined in inhomogeneous patches)

Near-Planckian regime ($\lambda \ll 1$)

$$\alpha \approx \lambda^2 - \frac{3r}{2(1+n)} \equiv \alpha_q \left( \frac{a}{\sqrt{\Delta}} \right)^{q\alpha}$$
Two regimes (well-defined in inhomogeneous patches)

Near-Planckian regime \( (\lambda \ll 1) \)

\[
\alpha \approx \lambda^2 \frac{3r}{2(1+n)} \equiv \alpha_q \left( \frac{a}{\sqrt{\Delta}} \right)^{q_\alpha}
\]

where

\[
q_\alpha = 4(1+n) - 3r, \quad \alpha_q = \left[ \frac{3\sqrt{3}}{2(1+n)} \frac{\Delta}{\Delta_{Pl}} \right]^{\frac{q_\alpha}{2(1+n)}}.
\]
Two regimes (well-defined in inhomogeneous patches)

Near-Planckian regime ($\lambda \ll 1$)

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$$q_\alpha = 4(1+n) - 3r, \quad \alpha_q = \left[ \frac{3\sqrt{3}}{2(1+n)} \frac{\Delta}{\Delta_{Pl}} \right]^{\frac{q_\alpha}{2(1+n)}}.$$ 

$$1 < q_\alpha < 6, \quad 1.6 \approx \frac{3^{3/4}}{\sqrt{2}} < \alpha_q < \frac{27}{4} \approx 6.8.$$
Coefficients

α maintains the same structure in different quantization schemes, where $c$ and $q$ are robust in the choice of the parameters.

'Natural' values (dictated by the form of the Hamiltonian or other considerations)

$c = 6$, $\alpha_c = 0$, $q = \sqrt{3}$,
Coefficients

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2. ‘Natural’ values (dictated by the form of the Hamiltonian or other considerations)

$$ r = 1, \quad n = 1/2 $$

$$ c = 6, \quad \alpha_c = 0, \quad \alpha_q = \sqrt{3}, \quad q_\alpha = 3. $$
Tensor perturbations
Bojowald-Hossain 2007

| Triad and connection separated into a FRW background and an inhomogeneous perturbation: |
| E\_a\_i = a^2 \delta E\_a\_i + \delta E\_a\_i, |
| A\_i\_a = c \delta a\_i\_a + (\delta \Gamma\_i\_a + \gamma \delta K\_i\_a) |

Then

\[ \delta E\_a\_i = -\frac{1}{2} a^2 h\_a\_i, \delta K\_i\_a = \frac{1}{2} (\frac{1}{\alpha} \partial \tau h\_a\_i + c \gamma h\_a\_i) \]

and

\[ \{ \delta K\_i\_a (x), \delta E\_b\_j (y) \} = 8 \pi G \delta b\_a \delta i\_j \delta (x, y) \]
Tensor perturbations
Bojowald-Hossain 2007

\[ ds^2 = -dt^2 + a^2 (\delta_{ij} + h_{ij}) dx^i dx^j \]
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and

\[ \{ \delta K^i_a(x), \delta E^b_j(y) \} = 8\pi G \delta^b_a \delta^i_j \delta(x, y) \]
Mukhanov equation

\[ \tau \equiv \int \frac{dt}{a}. \]

Only inverse-volume corrections:

\[ \partial_2 \tau h_{kk} + H(2 - d \ln \alpha d \ln a) \partial \tau h_{kk} + \alpha^2 k^2 h_{kk} = 0. \]

We solve it in large- and small-volume regimes separately.
Mukhanov equation

Conformal time \( \tau \equiv \int \frac{dt}{a} \).
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$$\partial^2_{\tau} h_k + \mathcal{H} \left( 2 - \frac{d \ln \alpha}{d \ln a} \right) \partial_{\tau} h_k + \alpha^2 k^2 h_k = 0.$$
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We solve it in large- and small-volume regimes separately.
Background

Tensor perturbations

\[ a = \tau_p, \quad H \equiv \partial_\tau a = aH = p\tau. \]

First slow-roll parameter

\[ \epsilon = -\frac{\dot{H}}{H^2} = 1 + \frac{1}{p}. \]

Inflation occurs for \( p < -1 \) (de Sitter: \( p = -1 \)), superinflation when \( -1 < p < 0 \).
\[ a = \tau^p, \quad \mathcal{H} \equiv \frac{\partial \tau a}{a} = aH = \frac{p}{\tau}. \]
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Outline

1. Background

2. Tensor perturbations
   - Near-Planckian regime
   - Quasi-classical regime
Near-Planckian regime: Solution

Mukhanov variable $w_k \equiv a_h k$, time variable $z \equiv \int d\tau \alpha = \tau \alpha / (1 + pq \alpha)$

$$\partial_z^2 w_k + \left( k^2 - 4 \nu^2 - \frac{1}{4} z^2 \right) w_k = 0,$$
where $\nu \equiv \frac{1}{2} - \frac{p}{(1 + pq \alpha)}$.

Solution:

$$w_k = C_1 \sqrt{-kz} \frac{H(1)}{\nu(-kz)} + C_2 \sqrt{-kz} \frac{H(2)}{\nu(-kz)}$$

$C_2 = 0$ (advancing plane wave at small scales)

Large- and short-wavelength limits of the solution ($\nu > 0$)

$$w_k \sim -\frac{iC_1}{2} \nu \Gamma(\nu) \pi \frac{1}{2 - \nu} \left| -kz \right| \ll 1,$$

$$w_k \sim C_1 \sqrt{2\pi} e^{-i \left( kz + \frac{\pi}{2} \nu + \frac{\pi}{4} \right)} \left| -kz \right| \gg 1.$$
Near-Planckian regime: Solution

- Mukhanov variable \( w_k \equiv ah_k \), time variable \( z \equiv \int d\tau \alpha = \tau \alpha/(1 + pq_\alpha) \)
Near-Planckian regime: Solution

- Mukhanov variable $w_k \equiv a h_k$, time variable $z \equiv \int d\tau \alpha = \tau \alpha/(1 + p q \alpha)$

- $\partial_z^2 w_k + \left( k^2 - \frac{4\nu^2 - 1}{4z^2} \right) w_k = 0$, where $\nu \equiv 1/2 - p/(1 + p q \alpha)$
Near-Planckian regime: Solution

- Mukhanov variable \( w_k \equiv ah_k \), time variable \( z \equiv \int d\tau \alpha = \tau \alpha/(1 + pq\alpha) \)
- \( \partial_z^2 w_k + \left( k^2 - \frac{4\nu^2 - 1}{4z^2} \right) w_k = 0 \), where \( \nu \equiv 1/2 - p/(1 + pq\alpha) \)
- Solution: \( w_k = C_1 \sqrt{-kz} H_\nu^{(1)}(-kz) + C_2 \sqrt{-kz} H_\nu^{(2)}(-kz) \)
Near-Planckian regime: Solution

- Mukhanov variable $w_k \equiv ah_k$, time variable $z \equiv \int d\tau \alpha = \tau \alpha / (1 + pq\alpha)$
- $\partial^2_z w_k + \left( k^2 - \frac{4\nu^2 - 1}{4z^2} \right) w_k = 0$, where $\nu \equiv 1/2 - p/(1 + pq\alpha)$
- Solution: $w_k = C_1 \sqrt{-kz} H_\nu^{(1)}(-kz) + C_2 \sqrt{-kz} H_\nu^{(2)}(-kz)$
- $C_2 = 0$ (advancing plane wave at small scales)
Near-Planckian regime: Solution

- Mukhanov variable $w_k \equiv a h_k$, time variable 
  $z \equiv \int d\tau \alpha = \tau \alpha / (1 + pq \alpha)$
- $\partial_z^2 w_k + \left(k^2 - \frac{4\nu^2 - 1}{4z^2}\right) w_k = 0$, where $\nu \equiv 1/2 - p/(1 + pq \alpha)$
- Solution: $w_k = C_1 \sqrt{-kz} H^{(1)}_{\nu}(−kz) + C_2 \sqrt{-kz} H^{(2)}_{\nu}(−kz)$
- $C_2 = 0$ (advancing plane wave at small scales)
- Large- and short-wavelength limits of the solution ($\nu > 0$)

$$w_k \sim -iC_1 \frac{2^\nu \Gamma(\nu)}{\pi} (-kz)^{1/2-\nu}, \quad |kz| \ll 1,$$

$$w_k \sim C_1 \sqrt{\frac{2}{\pi}} e^{-i(kz + \frac{\pi}{2} \nu + \frac{\pi}{4})}, \quad |kz| \gg 1.$$
Near-Planckian regime: Normalization

\[ C_1 \text{ is determined by choosing the Bunch–Davis vacuum, } \omega_k \sim e^{-ikz/\sqrt{2k}}. \]

Operator \[ \hat{u}_k = a \hat{h}_k = \omega_k a_k + \omega_k^* a_k^\dagger \] obeys

\[ [\hat{u}_k, \partial_\tau \hat{u}_k^\dagger] = \frac{32\pi \ell^2_{Pl}}{\alpha \delta(k_1, k_2)}. \]

Wronskian:

\[ \omega_k \partial_\tau \omega_k^* - \omega_k^* \partial_\tau \omega_k = i (32\pi \ell^2_{Pl})^{\alpha}. \]

Plugging in the short-scale solution, one gets

\[ |C_1| = \sqrt{8\pi^2 \ell^2_{Pl}/k}. \]
Near-Planckian regime: Normalization

Constant $C_1$ is determined by choosing the Bunch–Davis vacuum, $w_k \sim e^{-ikz}/\sqrt{2k}$. 

\[ \hat{u}_k = \hat{a}^\dagger_h k = w_k a_k + w_k^* a_k^\dagger \] 

\[ \left[ \hat{u}_k, \partial_\tau \hat{u}_{k'} \right] = \frac{32\pi\ell_s^2}{\alpha (k)} \left( \delta_{k, k'} \right) \] 

Wronskian: 

\[ w_k \partial_\tau w_k^* - w_k^* \partial_\tau w_k = i \left( \frac{32\pi\ell_s^2}{\alpha} \right) \] 

Plugging in the short-scale solution, one gets 

\[ |C_1| = \sqrt{\frac{8\pi}{2\ell_s^2}} k \]
Near-Planckian regime: Normalization

Constant $C_1$ is determined by choosing the Bunch–Davis vacuum, $w_k \sim e^{-ikz}/\sqrt{2k}$.

Operator $\hat{u}_k = a\hat{h}_k = w_ka_k + w^*_ka_k^\dagger$ obeys

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Plugging in the short-scale solution, one gets $|C_1| = \sqrt{8\pi^2 \ell_{\text{Pl}}^2/k}$. 
Near-Planckian regime: Spectrum

Horizon crossing defined when perturbations freeze: 
\[ k^* = \sqrt{4 \nu^2 - 1} \]

Well defined only if 
\[ p > \frac{1}{2(1 - q^\alpha)} \]

Stronger condition: 
\[ p > -\frac{1}{q^\alpha} > -1 \]

Modes exit horizon: The tensor spectrum is given by 
\[ A_{2T} = \frac{k^{3/2}}{a^2} \sum \propto \frac{H^2}{1 - \frac{pq^\alpha}{1 + pq^\alpha}} \]

Tensor spectral index: 
\[ n_T = 2(\epsilon + q^\alpha) \]
Near-Planckian regime: Spectrum

Horizon crossing defined when perturbations freeze:

\[ k_\ast = \frac{\sqrt{4\nu^2 - 1}}{2z} = \frac{\mathcal{H}}{\alpha} \sqrt{1 - q_\alpha - \frac{1}{p}}. \]
Near-Planckian regime: Spectrum

**Horizon crossing** defined when perturbations freeze:

\[ k_* = \frac{\sqrt{4\nu^2 - 1}}{2z} = \frac{\mathcal{H}}{\alpha} \sqrt{1 - q\alpha - \frac{1}{p}}. \]

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Near-Planckian regime: Spectrum

Horizon crossing defined when perturbations freeze:

\[ k_* = \frac{\sqrt{4\nu^2 - 1}}{2z} = \frac{\mathcal{H}}{\alpha} \sqrt{1 - q\alpha - \frac{1}{p}}. \]

Well defined only if \( p > 1/(1 - q\alpha) \). Stronger condition \( p > -1/q\alpha > -1 \) (\( z \) flows along the same direction at \( \tau \), modes exit horizon)
Near-Planckian regime: Spectrum

Horizon crossing defined when perturbations freeze:

\[ k_\star = \frac{\sqrt{4\nu^2 - 1}}{2z} = \frac{\mathcal{H}}{\alpha} \sqrt{1 - q_\alpha - \frac{1}{p}}. \]

Well defined only if \( p > \frac{1}{1 - q_\alpha} \). Stronger condition \( p > -\frac{1}{q_\alpha} > -1 \) (\( z \) flows along the same direction at \( \tau \), modes exit horizon)

Tensor spectrum:

\[
A_T^2 \equiv \frac{\mathcal{P}_h}{100} \equiv \frac{k^3}{200\pi^2 a^2} \sum_{+,\times} \left< |\hat{u}_k| H^2 \right> \bigg|_{k=k_\star}
\]
Near-Planckian regime: Spectrum

Horizon crossing defined when perturbations freeze:

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\[ \propto \frac{H^2}{\alpha^2} \]
Near-Planckian regime: Spectrum

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\[ \propto \frac{H^2}{\alpha^2} \propto k^2(1+p+pq\alpha)/(1+pq\alpha) \]
Near-Planckian regime: Spectrum

Horizon crossing defined when perturbations freeze:

\[ k_\ast = \frac{\sqrt{4v^2 - 1}}{2z} = \frac{\mathcal{H}}{\alpha} \sqrt{1 - q\alpha - \frac{1}{p}}. \]

Well defined only if \( p > 1/(1 - q\alpha) \). Stronger condition \( p > -1/q\alpha > -1 \) (\( z \) flows along the same direction at \( \tau \), modes exit horizon)

Tensor spectrum:

\[
A_T^2 \equiv \frac{\mathcal{P}_h}{100} \equiv \frac{k^3}{200\pi^2a^2} \sum_{+,\times} \left\langle |\hat{u}_k \ll \mathcal{H}|^2 \right\rangle \bigg|_{k=k_\ast}
\]

\[
\propto \frac{H^2}{\alpha^2} \propto k^{2(1+p+pq\alpha)/(1+pq\alpha)}
\]

Tensor spectral index:

\[
n_T \equiv \frac{d \ln A_T^2}{d \ln k} \bigg|_{k=k_\ast} = \frac{2(\epsilon + q\alpha)}{\epsilon + q\alpha - 1}
\]
Stochastic background of primordial gravitational waves

\[ \Omega_{gw} = \frac{1}{\rho_{\text{crit}}} \frac{d\rho_{gw}}{d \ln f} \propto T(k)^2 A_T^2 \]
Stochastic background of primordial gravitational waves

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\[ n_T \approx \frac{1}{\ln f - \ln f_0} \ln \left( 2.29 \times 10^{14} \frac{h^2 \Omega_{gw}(f)}{r} \right) \]
Stochastic background of primordial gravitational waves

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- Pulsar timing, LIGO, LISA, BBN place strong constraints.
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- Pulsar timing, LIGO, LISA, BBN place strong constraints.
- Taking upper bound \( r < 0.30 \), from pulsar timing \( n_T \lesssim 0.79 \), from BBN \( n_T \lesssim 0.15 \).
Stochastic background of primordial gravitational waves

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- Pulsar timing, LIGO, LISA, BBN place strong constraints.
- Taking upper bound \( r < 0.30 \), from pulsar timing \( n_T \lesssim 0.79 \), from BBN \( n_T \lesssim 0.15 \).
- If \( r \sim 10^{-8} \), still these bounds are \( n_T < 1 \).
Near-Planckian regime: Excluded?

- Quasi-de Sitter limit ($\epsilon \approx 0$):
  \[ n_T \approx \frac{2q^\alpha}{(q^\alpha - 1)} > \frac{12}{5}. \]
  Strong blue tilt.

- Deep superacceleration ($\epsilon \ll -q^\alpha$):
  \[ n_T \approx 2. \]
  Strong blue tilt.

Near-Planckian phase might have occurred only at very early times (unobservably large scales) and for a short period.

Scale-invariant or red-tilted tensor spectrum achieved in the interval

\[ -\frac{1}{q^\alpha} < p \ll -\frac{1}{(q^\alpha + 1)}, \]

but could spoil scale invariance of scalar spectrum.

1: $r$ could be fine tuned to be small but scalar sector not available.

2: Anomaly cancellation does not happen in scalar sector in this regime, which may be a sign that perturbation theory fails to converge.

3: Close to the bounce, power-law evolution may not be a good approximation. However, $a \approx \text{const.}$
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Outline

1. Background

2. Tensor perturbations
   - Near-Planckian regime
   - Quasi-classical regime
Quasi-classical regime: Solution

\begin{equation}
\frac{\partial^2 \tau}{\partial \tau^2} w_k + c H (\alpha - 1) \frac{\partial \tau}{\partial \tau} w_k + \left\{ (2 \alpha - 1) k^2 + H^2 \left[ \epsilon - 2 - c (\alpha - 1) \right] \right\} w_k \approx 0.
\end{equation}

Solution perturbative in $\alpha$:

\begin{equation}
w_k = w_k(0) + \alpha c w_k(1)\frac{\partial^2 \tau}{\partial \tau^2} w_k(0) + \left\{ k^2 + H^2 (\epsilon - 2) \right\} w_k(0) = 0,
\end{equation}

\begin{equation}
\frac{\partial^2 \tau}{\partial \tau^2} w_k(1) + \left\{ k^2 + H^2 (\epsilon - 2) \right\} w_k(1) + r(\tau) = 0,
\end{equation}

where $r(\tau) \equiv \left( \sqrt{\Delta a} \right) c H \frac{\partial \tau}{\partial \tau} w_k(0) + \left( 2 k^2 - c H^2 \right) w_k(0)$. 
Quasi-classical regime: Solution

Mukhanov equation:

\[ \partial^2_{\tau} w_k + c \mathcal{H}(\alpha - 1) \partial_{\tau} w_k + \{(2\alpha - 1)k^2 + \mathcal{H}^2[\epsilon - 2 - c(\alpha - 1)]\}w_k \approx 0. \]
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Solution perturbative in \( \alpha_c \) \( (\alpha_c \neq 0; \text{natural choice trivial}) \):

\[ w_k = w_k^{(0)} + \alpha_c w_k^{(1)} \]
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\partial_\tau^2 w_k + c \mathcal{H} (\alpha - 1) \partial_\tau w_k + \{ (2\alpha - 1) k^2 + \mathcal{H}^2 [\epsilon - 2 - c (\alpha - 1)] \} w_k \approx 0.
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\]

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\partial_\tau^2 w_k^{(1)} + [ k^2 + \mathcal{H}^2 (\epsilon - 2) ] w_k^{(1)} + r(\tau) = 0,
\]

\[
r(\tau) \equiv \left( \frac{\sqrt{\Delta}}{a} \right)^c \left[ c \mathcal{H} \partial_\tau w_k^{(0)} + (2k^2 - c \mathcal{H}^2) w_k^{(0)} \right]
\]
Quasi-classical regime: Asymptotic solutions
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At large scales:

\[ w_k \ll H = C_1 (1 + \alpha c C_2) \tau^p \]
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Quasi-classical regime: Asymptotic solutions

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\[ w_{k \gg \mathcal{H}} = w_{k \gg \mathcal{H}}^{(0)} \left[ 1 + \alpha_c \frac{i k \tau}{c p - 1} \left( \frac{\sqrt{\Delta}}{\tau^p} \right)^c \right] \]
Quasi-classical regime: Normalization
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\[
C_1(k) = \sqrt{\frac{16\pi\ell_P^2}{k}} \frac{e^{-ik\tau_*}}{\tau_*^p} \equiv \tilde{C}_1 k^{p - 1/2}\\
C_2(k) = \frac{ik_*\tau_*}{cp - 1} \left( \frac{\sqrt{\Delta}}{\tau_*^p} \right)^c \equiv \tilde{C}_2 k^{cp}
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- Correction term decays in time.
Quasi-classical regime: Spectrum

\[ A^2 T = 25 \pi k^2 \left( 1 + p \right) p \left( p - 1 \right) \left( 1 + \delta P \right), \]

where \( \delta P \equiv \alpha^2 c | \tilde{C}^2 |^2 k^2 c p \)

Tensor index:

\[ n_T \approx 2 \left( 1 + p + c \delta P \right) = -2 \left( \epsilon + c \delta P \right)^{-1} \]
Quasi-classical regime: Spectrum

\[ A^2_T = \frac{4 \ell_{Pl}^2}{25\pi} \frac{k^{2(1+p)}}{[p(p - 1)]^p} (1 + \delta_{Pl}), \]
Quasi-classical regime: Spectrum

\[ A_T^2 = \frac{4\ell_{Pl}^2}{25\pi} \frac{k^{2(1+p)}}{[p(p - 1)]^p} (1 + \delta_{Pl}), \]

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Tensor index:

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Conclusions

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Conclusions

- Near-Planckian regime possibly disfavoured.
- However, there are caveats to be addressed.
- Only nonperturbative formalisms (covariant, $\delta N$, separate universe, etc.) could be trusted (also relevant for anomaly issue).
- Quasi-classical result reliable, but scalar sector still under inspection.