Generalization of Vasiliev Higher Spin Theory with an Action Principle

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w/ Nicolas Boulanger and Per Sundell, arXiv:1505.04957

Higher Spin Workshop, Penn State, August 29, 2015
Motivations for this work

• How unique is Vasiliev’s theory?

Extensions beyond Chan-Paton factors, supersymmetry and parity breaking interaction ambiguity with enlarged the HS algebra?

• Higher spin invariants of relevance to the computation of tree-level n-point amplitudes have been constructed. However, considered as part of an effective action, their coefficients are arbitrary.

Can an extension of Vasiliev theory remove this arbitrariness?

• Can prospects for construction of an action be improved?

We will see that there is a remarkable extension with desired properties, leading to non-commutative Frobenius-Chern-Simons gauge theory.
Bosonic Vasiliev theory in a nutshell

- Formulated in terms of differential forms on $\mathcal{M}_4 \times \mathbb{Z}_4$ with coordinates $(x^\mu, z^\alpha, \bar{z}^{\dot{\alpha}})$ and exterior derivative $d = dx + dz + d\bar{z}$ where $Z^\alpha = (z^\alpha, \bar{z}^{\dot{\alpha}})$ are Grassmann even and non-commutative $SL(2, C)$ spinors.

- In the case of the minimal bosonic models, consisting of integer spins, the higher-spin representation content can be encoded using a second set of Grassmann even non-commutative $SL(2, C)$ spinors $Y^\alpha = (y^\alpha, \bar{y}^{\dot{\alpha}})$.

- Introduce one-form $W(x, Z; Y)$ and zero-form $\Phi(x, Z; Y)$.

- In the normal ordering of the Weyl algebra

$$[Y^\alpha, Y^\beta]_* = -[Z^\alpha, Z^\beta]_* = 2iC^V_{\alpha\beta}, \quad [Y^\alpha, Z^\beta]_* = 0$$

with respect to $Y^\alpha \pm Z^\alpha$, symbol composition reads

$$P(Z, Y) * Q(Z, Y) = \int_{U, V} P(Z + U, Y + U) Q(Z - V, Y + V) e^{-iU \cdot V}$$
• Twisted central two-form

\[ J = -\frac{i}{4} \left( dz^\alpha dz_\alpha \kappa + d\bar{z}^{\dot{\alpha}} d\bar{z}_{\dot{\alpha}} \bar{\kappa} \right), \]

defined in terms of the inner Klein operator

\[ \kappa = e^{iy^\alpha z_\alpha}, \quad \kappa \star \kappa = 1. \]

• By its definition

\[ dJ = 0, \quad J \ast W = \pi(W) \star J, \quad J \ast \Phi = \pi(\Phi) \star J, \]

where the star product outer automorphisms

\[ \pi(y, \bar{y}, z, \bar{z}) = (-y, \bar{y}, -z, \bar{z}), \quad d \circ \pi = \pi \circ d \]

• Real integer spin fields

\[ \pi\bar{\pi}(W, \Phi, J) = (W, \Phi, J), \quad (W^\dagger, \Phi^\dagger, J^\dagger) = (-W, \pi(\Phi), -J) \]
• Vasiliev equations

\[ dW + W \star W + \Phi \star (J - \bar{J}) = 0 , \]
\[ d\Phi + W \star \Phi - \Phi \star \pi(W) = 0 . \]

• Cartan integrable, i.e. consistent with \( d^2 = 0 \), hence invariant under

\[ \delta W = d\epsilon + [W, \epsilon] \star , \quad \delta \Phi = -\epsilon \star \Phi + \Phi \star \pi(\epsilon) , \]

which define adjoint and twisted-adjoint representations, respectively.

• Spin \( s \geq 1 \) gauge fields arise in \( W|_{Z=0} \) while massless scalar field and massless spin \( s \geq 1 \) Weyl tensors are packed into in \( \Phi|_{Z=\bar{y}=0} \).

• The adjoint two-form \( \Phi \star (J - \bar{J}) \), which can be replaced by a more general interaction ambiguity (provided parity symmetry is yielded), deforms the symplectic structure on \( \mathcal{Z}_4 \).

Questioning the rational behind \( J \) being non-dynamical is the key consideration for the generalization that we have found.
Extension of Vasiliev theory

- In addition to $M_4 \times Z_4$, introduce an auxiliary dimension, such that the total manifold on which the fields live is

$$M_9 = X_5 \times Z_4$$

where $\partial Z_4 = \emptyset$ and $\partial X_4$ contains $M_4$. [Boulanger and Sundell, 2011].

(In the spirit of Kazinski, Lyakhovich and Sharapov, 2005).

- All fields valued in associative higher spin algebra $\mathcal{A}$ extended by inner and outer Klein operators with trace $\text{Tr}_A$.

- The algebra $\mathcal{A}$ star commutes to the algebra $\Omega(Z_4)$ of forms on $Z_4$:

$$\mathcal{A} \star \mathcal{O}(Z_4) = \mathcal{O}(Z_4) \star \mathcal{A}.$$ 

- Comparison with Vasiliev’s model requires taking

$$\pi_y \pi_y (\Phi, W) = \pi_z \pi_z (\Phi, W) = (\Phi, W),$$

which can indeed be imposed perturbatively and for the exact solutions of importance.

⇝ Formalism capable of incorporating more general noncommutative base manifolds.
Bifundamental master fields

- Introduce a zero-form $B$, a pair of one-forms $(A, \tilde{A})$ and a two-form $\tilde{B}$ on $\mathcal{M}_9$ such that

$$\Phi \in B|_{\partial \mathcal{M}_9} , \quad W \in \frac{1}{2}(\tilde{A} + A)|_{\partial \mathcal{M}_9} ,$$

$$\Phi \star (J - \bar{J}) \in (B \star \tilde{B})|_{\partial \mathcal{M}_9} .$$

- Let $B$ transform in a bifundamental representation of $\mathcal{A}$ (more below)

$$\delta B = -\epsilon \star B + B \star \tilde{\epsilon} ,$$

- Let $\tilde{B}$ transform in the opposite bifundamental representation

$$\delta \tilde{B} = -\tilde{\epsilon} \star \tilde{B} + \tilde{B} \star \epsilon .$$

\[ \leadsto \text{Drastic reduction of the number of HS invariants.} \]

- Choose boundary conditions on $\mathcal{Z}_4$ such that

$$\tilde{B} = f(B, J, \bar{J}) ,$$

$\tilde{B}$ plays the rôle of a dynamical counter part of $J$ and $\bar{J}$. 
Bifundamental master field equations

Postulate the following Cartan integrable system:

\[ dA + A \star A - B \star \tilde{B} = 0 , \]
\[ d\tilde{A} + \tilde{A} \star \tilde{A} - \tilde{B} \star B = 0 , \]
\[ dB + A \star B - B \star \tilde{A} = 0 , \]
\[ d\tilde{B} + A \star \tilde{B} - \tilde{B} \star A = 0 . \]

Gauge transformations:

\[ \delta A = d\epsilon + [A, \epsilon]_* + \eta \star \tilde{B} + B \star \tilde{\eta} , \]
\[ \delta \tilde{A} = d\tilde{\epsilon} + [\tilde{A}, \tilde{\epsilon}]_* + \tilde{\eta} \star B + \tilde{B} \star \eta , \]
\[ \delta B = d\eta + A \star \eta + \eta \star \tilde{A} - \epsilon \star B + B \star \tilde{\epsilon} , \]
\[ \delta \tilde{B} = d\tilde{\eta} + \tilde{A} \star \tilde{\eta} + \tilde{\eta} \star A - \tilde{\epsilon} \star \tilde{B} + \tilde{B} \star \epsilon . \]
Frobenius algebra

Finite-dimensional, unital, associative algebra $A$ defined over a field $k$, and equipped with a non-degenerate bilinear form $\sigma : A \times A \rightarrow k$ that satisfies the equation $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$. For example, any matrix algebra with the bilinear form $\sigma(a, b) = \text{tr}(a \cdot b)$ is a Frobenius algebra.

- Employ an eight-dimensional associative algebra $\mathcal{F}$ with elements $(e_{ij}, h e_{ij})$ where $e_{ij}$ are $2 \times 2$ matrices with nonvanishing entry 1 only at the $i$th row and $j$th column, and $h$ is an outer Kleinian element satisfying

$$[h, e_{11}] = 0 = [h, e_{22}] , \quad \{h, e_{12}\} = 0 = \{h, e_{12}\}$$

- Frobenius algebra trace operation is defined as

$$\text{Tr}_{\mathcal{F}} \sum_{i,j} e_{ij} M^{ij}(h) = M^{11}(0) + M^{22}(0)$$
Assembling the master fields as

\[ X = \sum_{i,j} X^{ij} e_{ij} = \begin{pmatrix} A & B \\ \tilde{B} & \tilde{A} \end{pmatrix} \]

the field equations read

\[ F_X := dX + hX h \star X = 0 , \]

or equivalently

\[ (hd)(hX) + (hX) \star (hX) = 0 . \]
Inside the bulk of $\mathcal{M}_9$, introduce Lagrange multiplier master fields

$$P = \sum_{i,j} P^{ij} e_{ij} = \begin{pmatrix} V & U \\ \tilde{U} & \tilde{V} \end{pmatrix}$$

Duality extended field content

$$\deg(B, A, \tilde{A}, \tilde{B}) \in \{(2n, 1 + 2n, 1 + 2n, 2 + 2n)\}_{n=0,1,2,3},$$

$$\deg(\tilde{U}, V, \tilde{V}, B) = \{(8 - 2n, 7 - 2n, 7 - 2n, 6 - 2n)\}_{n=0,1,2,3}.$$
Superconnection and Frobenius-Chern-Simons action

Assemble $X$ and $P$ into superconnection

$$Z = hX + P$$

Use the trace operation $\text{Tr}_{\mathcal{F} \otimes \mathcal{A}}$ (more on $\mathcal{A}$ and $\text{Tr}_\mathcal{A}$ below) to construct the action

$$S_{\text{FCS}} = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{F} \otimes \mathcal{A}} \left( \frac{1}{2} Z \star qZ + \frac{1}{3} Z \star Z \star Z \right) , \quad q := hd .$$

Globally defined generalized Hamiltonian action

$$S_H = S_{\text{FCS}} - \frac{1}{4} \int_{\partial \mathcal{M}_9} \text{Tr}_{\mathcal{F} \otimes \mathcal{A}} [h\pi_h(Z) \star Z]$$

$$= \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{F} \otimes \mathcal{A}} \left( P \star F^X + \frac{1}{3} P \star P \star P \right) ,$$

where $q := hd$ and $\pi_h(h) = -h$, and assuming structure group is associated to $X$ and that $P|_{\partial \mathcal{M}_9} = 0$.
The component form of the action:

\[
S_H = \int_{\mathcal{M}_9} \text{Tr}_A \left[ V \star \left( F + B \star \tilde{B} \right) + \tilde{U} \star DB + \frac{1}{3} V^3 - V \star U \star \tilde{U} \\
+ \tilde{V} \star \left( \tilde{F} + \tilde{B} \star B \right) + U \star \tilde{D}\tilde{B} + \frac{1}{3} \tilde{V}^3 - \tilde{V} \star \tilde{U} \star U \right],
\]

- The terms cubic in Lagrange multipliers require the duality extended field content.

- Using the boundary condition \( P|_{\partial \mathcal{M}_9} = 0 \), the field equation reduce to \( dA + A \star A + B \star \tilde{B} = 0 \) etc but note that these are duality extended.
Base manifold

- The topology of $\mathcal{Z}_4$ may be chosen in a variety ways as to achieve classes of symbols with well-defined star products and finite integrals.

- We choose

$$\mathcal{Z}_4 = S^2 \times S^2$$

obtained from $\mathbb{R}^4$ by adding suitable points at infinity to $\mathbb{R}^4$.

- For $\mathcal{X}_5$, one simple alternative is the semi-infinite cylinder

$$\mathcal{X}_5 = \mathcal{X}_4 \times [0, \infty) , \quad \partial \mathcal{X}_4 = \emptyset .$$

though multiple boundaries may of course ultimately be much more interesting.

The key point is that the topology of $\mathcal{M}_9$ is now part of the moduli space of the theory, and the Poisson structures on it is deformed by the vacuum expectation values of $B \star \tilde{B}$ and $\tilde{B} \star B$. 
The higher spin algebra

\[ \mathcal{A} = \Pi_+ \star \bigoplus_{m, \bar{m} = 0, 1} \mathcal{P}_{m, \bar{m}}(y, \bar{y}, k_y, \bar{k}_{\bar{y}}) \star (\kappa_y)^m \star (\bar{\kappa}_{\bar{y}})^{\bar{m}} \star \Pi_+ , \]

\[ \Pi_\pm = \frac{1}{2} (1 \pm k_y \star \bar{k}_{\bar{y}}) \]

- \( \mathcal{P}_{m, \bar{m}} \) consist of arbitrary polynomials in oscillators \((y_\alpha, \bar{y}_\dot{\alpha})\) and outer Klein operators \((k_y, \bar{k}_{\bar{y}})\). (Will comment later on suitable non-polynomial extension).

- The \( \Pi_+ \) projection removes odd polynomials in \((y, \bar{y})\) leading to a model consisting (perturbatively) of Lorentz tensorial higher spin fields.

- Inner Klein operators for the Weyl algebra

\[ \kappa_y = 2\pi \delta^2(y) \quad \bar{\kappa}_{\bar{y}} = 2\pi \delta^2(\bar{y}) \]

are required to generate the vacuum expectation values for \( \tilde{B} \).
• Composition rule

\[
P_1 \star P_2 = \int_{\mathbb{R}^4} \frac{d^2 \xi d^2 \tilde{\xi} d^2 \eta d^2 \tilde{\eta}}{(2\pi)^4} e^{i(\eta^\alpha \xi_\alpha + \tilde{\eta}^\alpha \tilde{\xi}_\alpha)} \times \]

\[
\times \quad P_1(y + \xi, \bar{y} + \tilde{\xi}; k_y, \bar{k}_{\bar{y}}) P_2(y + \eta, \bar{y} + \tilde{\eta}; k_y, \bar{k}_{\bar{y}})
\]

realizes the Weyl algebra of the oscillator algebra

\[
[y^\alpha, y^\beta]_* = 2i\epsilon^{\alpha\beta}, \quad [\bar{y}^\dot{\alpha}, \bar{y}^\dot{\beta}]_* = 2i\epsilon^{\dot{\alpha}\dot{\beta}},
\]

\[
\{k_y, y^\alpha\}_* = 0 = \{\bar{k}_{\bar{y}}, \bar{y}^\dot{\alpha}\}_*, \quad k_y \star k_y = \bar{k}_{\bar{y}} \star \bar{k}_{\bar{y}} = 1.
\]

in terms of Weyl ordered symbols.
Trace operations and variational principle

• The trace operation $\text{Tr}_A$ is defined as

$$\text{Tr}_A P(y, \bar{y}; k_y, \bar{k}_y) = P_{00}(0, 0; 0, 0)$$

using symbols defined in the Weyl order.

• Recall the Frobenius algebra trace operation

$$\text{Tr}_\mathcal{F} \sum_{i,j} e_{ij} M^{ij}(h) = M^{11}(0) + M^{22}(0)$$

• General variation of the action:

$$\delta S = \int_{\mathcal{M}_9} \text{Tr}_{\mathcal{F} \otimes A} \left( \delta X \star h R^P h + \delta P \star R^X + d(\delta X \star P) \right)$$
Assuming that $X$ is free to fluctuate at $\partial M_9$, the variational principle thus implies that

$$R^X := dX + hX h \star X + P \star P = 0$$

$$R^P := dP + X \star P + hPh \star X = 0$$

$$P |_{\partial M_9} = 0$$

The bulk action is invariant under $\epsilon^X$ transformation, while under $\epsilon^P$ transformation we get:

$$\delta_{\epsilon^P} S_H = \int_{M_9} \text{Tr}_{\mathcal{F} \otimes \mathcal{A}} d \left( \epsilon^P \star R^X \right)$$

This gives the condition

$$\epsilon^P |_{\partial M_9} = 0$$
Reduction of the system

In order to more fully exhibit the topological degrees of freedom contained in the field $\tilde{B}$, we consider the following Ansatz for the duality extended system on $\mathcal{X}_4 \times \mathcal{Z}_4$:

\[
V = \tilde{V} = 0 = U = \tilde{U} , \quad \tilde{A} = A = W , \\
\tilde{B} = \sum_{r=0}^{N} \sum_{m, \bar{m} = 0, 1} \mathcal{W}_{r; m, \bar{m}}(B) \star (J)^{*m} \star (\bar{J})^{*\bar{m}} \star J_X^{(r)} ,
\]

where $J$ and $\bar{J}$ are the closed and central forms on $\mathcal{Z}_4$ defined above, $J_X^{(r)}$ is a basis for the even-degree de Rham cohomology on $\mathcal{X}_4$, which are central and closed elements, and $\mathcal{W}_{r; m, \bar{m}}$ are star functions. We assume that

\[
J_X^{(0)} = 1 , \quad \mathcal{W}_{0; 0, 0} = 0 .
\]

Substituting the above Ansatz into the field equations, we obtain

\[
F_W = - \sum_{r=0}^{N} \sum_{m, \bar{m} = 0, 1} B \star \mathcal{W}_{r; m, \bar{m}}(B) \star (J)^{*m} \star (\bar{J})^{*\bar{m}} \star J_X^{(r)} ,
\]

\[
D_W B = 0 ,
\]

where $F_W = dW + W \star W$ and $D_W B = dB + W \star B - B \star W$. 
Let us compare the above result, once written using normal order, with the recent proposal of Vasiliev in which the field equations, adapted to our notation, take the form

\[ F_W = \mathcal{V}(B) \star J - \mathcal{\overline{V}}(B) \star \overline{J} + \mathcal{U}(B) \star J \star \overline{J} + gJ \star \overline{J} + \mathcal{L} , \]
\[ D_W B = 0 , \]

where the star functions are polynomials vanishing at \( B = 0 \), \( g \) is a constant and

\[ \mathcal{L} = \mathcal{L}[2] + \mathcal{L}[4] , \]

belong to the de Rham cohomology on \( \mathcal{X}_4 \).
Adapting the reduced FCS system

Upon taking the only nonvanishing star functions to be $\mathcal{W}_{r;0,0}$ ($r \geq 1$), $\mathcal{W}_{0;1,0}(B)$, $\mathcal{W}_{0;0,1}(B)$ and $\mathcal{W}_{0;1,1}(B)$, and assuming that

$$\mathcal{W}_{0;1,0}(\nu) = 0 ,$$

one can redefine

$$B \rightarrow \nu + B ,$$

after which the reduced FCS equations take the form

$$F_W = - (\nu + B) \star \left[ \mathcal{W}_{0;1,0}(\nu + B) \star J + \mathcal{W}_{0;0,1}(\nu + B) \star \overline{J} + \mathcal{W}_{0;1,1}(\nu + B) \star J \star \overline{J} \right]$$

$$- \sum_{r=1}^{N} (\nu + B) \star \mathcal{W}_{r;0,0}(\nu + B) \star J_{X}^{(r)} ,$$

$$D_W B = 0 .$$
Thus, we identify the curvature deformations of Vasiliev’s recent duality system as a subset of the deformations above, with

\[ \mathcal{V}(B) = -(\nu + B) \star \mathcal{W}_{0;1,0}(\nu + B) , \quad \overline{\mathcal{V}}(B) = (\nu + B) \star \mathcal{W}_{0;0,1}(\nu + B) , \]
\[ \mathcal{U}(B) = -(\nu + B) \star \mathcal{W}_{0;1,1}(\nu + B) - g , \quad g = -\nu \mathcal{W}_{0;1,1}(0) , \]
\[ \mathcal{L} = -\nu \sum_{r} \mathcal{W}_{r;0,0}(\nu) J_{X}^{(r)} . \]

Compared with Vasiliev’s recent system, the above system also contains an extra non-central deformation given by

\[ -\sum_{r=1}^{N} (\nu + B) \star \mathcal{W}_{r;0,0}(\nu + B) \star J_{X}^{(r)} + \nu \sum_{r} \mathcal{W}_{r;0,0}(\nu) J_{X}^{(r)} , \]

the consequences of which remain to be investigated.
Comparison: Central terms versus boundary deformations

- The integral of $\mathcal{L}[2]$ evaluated on the Didenko–Vasiliev black hole solution has been interpreted as a “black hole charge”, and the integral of $\mathcal{L}[4]$ has been proposed as a candidate for the generating functional of correlators in the $AdS_4/CFT_3$ higher spin holography [Vasiliev’15].

- These interpretations, however, make use of non-standard gauge fixing procedures in which the globally defined gauge parameters associated with the de Rham cohomology elements are used to remove locally defined gauge fields, which is only possible if the latter belong to trivial bundles.

- Moreover, the direct connection between the charges of $\mathcal{L}$ and the partition function remains unclear in the absence of a path integral.
Instead, one may accommodate similar charges in an on-shell action derived from a path integral of AKSZ type, and without having to employ the gauge fixing procedure mentioned above, by adding Chern classes to the generalized Hamiltonian action:

\[ S_{\text{tot}} = S_H + \sum_{p,n} \int_{\mathcal{X}_{2p} \times \mathbb{Z}_4} \text{Tr}_A(x_n F^{*n} + \tilde{x}_n \tilde{F}^{*n}) , \]

where \( \mathcal{X}_{2p} \subseteq \mathcal{X}_4 \) (possibly augmented by Chern-Simons forms at boundaries).
Zero-form charges versus new HS invariants

- In the standard Vasiliev theory, a special classes of invariants are constructed as

\[
\mathcal{I}_{2n} = \int_{\mathbb{Z}_4} \text{Tr}_A (k_y \ast \bar{k}_{ij} \ast B^{(2n)} \ast J \ast \bar{J})|_{p_0}
\]

referred to as zero-form charges, as they are evaluated at a single point.

- These functionals have been shown [Colombo, Sundell, Didenko, Skvortsov] to be building blocks for higher spin amplitudes.

- They can be employed as deformations of a previously constructed [Boulanger and Sundell] Hamiltonian action for the original Vasiliev system. However, each building block comes with an arbitrary coefficient.

- In the FCS model, we have instead only two invariants that yield zero-form charges in Vasiliev branch. These are:

\[
\int_{x_0 \times \mathbb{Z}_4} \text{Tr}_A (B \ast \tilde{B})^n \quad \text{for} \quad n = 1, 2
\]

The consequences for amplitude computation under investigation.
Comments on perturbative analysis

- Define $W = (A + \tilde{A})/2$ and $K = (A - \tilde{A})/2$ and expand perturb about the vacuum solution

$$\tilde{B}^{(0)} = J - \bar{J}, \quad W^{(0)} = L^{-1} \star dL, \quad K^{(0)} = 0, \quad B^{(0)} = 0$$

we have shown that the theory does not give rise to any additional degrees of freedom under certain natural assumptions.

- Classical moduli arise in the solution for $\tilde{B}$, accounting for an interaction ambiguity in Vasiliev’s theory.
Comments on nonpolynomial extension of $A$ and one-body solutions

- The polynomial $P_{m, \bar{m}}$ we have in the HS algebra $A$ can be extended by inclusion of suitable nonpolynomials such that the trace operation still makes sense.

- This extension is associated with the tensor product of singleton and anti-singleton representation and gives rise to Coulomb/Schwarzschild-like charged objects (AdS stationary solutions). Iazeolla and Sundell'13

- These degrees of freedom arise also in Vasiliev’s theory even though they are usually not considered to be part of the perturbative spectrum.

Details and consequences are currently under investigation.
Open Problems

- Construction of the boundary deformations and techniques for their regularization.
- Quantization of the FCS model using TFT/AKSZ methods.
- Computation of amplitudes and making contact with CS-matter systems.
- Making contact with topological open strings and string partons arising in tensionless limit in AdS.